Regress+

Appendix A

A Compendium of Common Probability Distributions

Version 2.3

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PREFACE

This Appendix contains summaries of the probability distributions found in *Regress+*.

All distributions are shown in their parameterized, not standard forms. In some cases, the definition of a distribution may vary slightly from a definition given in the literature. This happens either because there is more than one definition or, in the case of parameters, because *Regress+* requires a parameter to be constrained, usually to guarantee convergence.

There are a few well-known distributions that are not included here, either because they are seldom used to model empirical data or they lack a convenient analytical form for the CDF. Conversely, many of the distributions that are included are rarely discussed yet are very useful for describing real-world datasets.

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September, 1999

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Distributions shown in **bold** are those which appear in the *Regress+* menu. Distributions shown in plain text are aliases and/or special cases.

Continuous Distributions

Continuous Mixtures

Discrete Distributions

Discrete Mixtures

Description of Included Items

Each of the distributions is described in a two-page summary. The summary header includes the distribution name and the parameter list, along with the numerical range for which variates and parameters (if constrained) are defined.

Each distribution is illustrated with at least one example. In this figure, the parameters used are shown in parentheses, in the order listed in the header. Expressions are then given for the PDF and CDF. Remaining subsections, as appropriate, are as follows:

Parameters

This is an interpretation of the meaning of each parameter, with the usual literature symbol (if any) given in parentheses.

Unless otherwise indicated, parameter A is a *location* parameter, positioning the overall distribution along the abscissa. Parameter B is a *scale* parameter, describing the extent of the distribution. Parameters C and, possibly, D are *shape* parameters which affect skewness, kurtosis, etc. In the case of binary mitures, there is also a *weight*, p, for the first component.

Moments, etc.

Provided that there are closed forms, the mean, variance, skewness, kurtosis, mode, median, first quartile (Q1), and third quartile (Q3) are described along with the quantiles for the mean (qMean) and mode (qMode). If random variates are computable with a closed-form expression, the latter is also given.

Note that the mean and variance have their usual units while the skewness and kurtosis are dimensionless. Furthermore, the kurtosis is referenced to that of a standard **Normal** distribution (kurtosis = 3).

Notes

These include any relevant constraints, cautions, etc.

Aliases and Special Cases

These alternate names are also listed as well in the Table of Contents.

Characterizations

This list is far from exhaustive. It is intended simply to convey a few of the more important situations in which the distribution is particularly relevant.

Obviously, so brief a account cannot begin to do justice to the wealth of information available. For fuller accounts, the aforementioned references, [KOT82], [JOH92], and [JOH94] are excellent starting points.

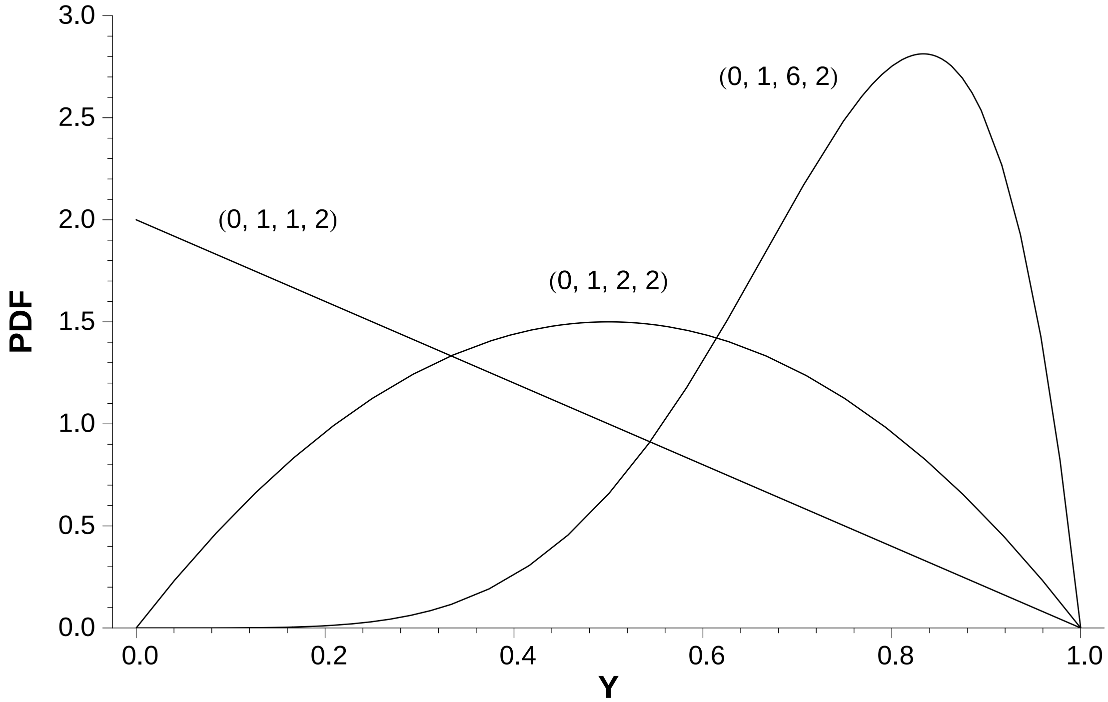
Legend

Shown below are definitions for some of the less common functions and abbreviations used in this Appendix.

int(y)	integer or floor function
Φ(z)	standard cumulative **Normal** distribution
erf(z)	error function
Γ(z)	complete Gamma function
Γ(w,x)	incomplete Gamma function
ψ(z), ψ'(z), etc.	diamma function and its derivatives
I(x,y,z)	(regularized, normalized) incomplete Beta function
H(n)	n^{th} harmonic number
ζ(s)	Riemann zeta function
$\binom{N}{k}$	number of combinations of N things taken k at a time
e	base of the natural logarithms = 2.71828...
γ	EulerGamma = 0.57721566...
u	a **Uniform**(0,1) random variate
i.i.d.	independent and identically distributed
iff	if and only if
N!	N factorial = N (N–1) (N–2) ... (1)
~	(is) distributed as

Beta(A,B,C,D)

$A < y < B, \quad C, D > 0$



$$\text{PDF} = \frac{\Gamma(C+D)}{\Gamma(C)\,\Gamma(D)\,(B-A)^{C+D-1}}\,(y-A)^{C-1}\,(B-y)^{D-1}$$

$$\text{CDF} = I\left(\frac{y-A}{B-A}, C, D\right)$$

Parameters -- A: Location, B: Scale (upper bound), C, D (p, q): Shape

Moments, etc.

$$\text{Mean} = \frac{A\,D + B\,C}{C+D}$$

$$\text{Variance} = \frac{C\,D\,(B-A)^2}{(C+D+1)\,(C+D)^2}$$

$$\text{Skewness} = \frac{2\,C\,D\,(D-C)}{(C+D)^3\,(C+D+1)\,(C+D+2)\left[\dfrac{C\,D}{(C+D)^2\,(C+D+1)}\right]^{\frac{3}{2}}}$$

$$\text{Kurtosis} = 3\left[\frac{\left(C^2 (D+2) + 2D^2 + C\,D\,(D-2)\right)(C+D+1)}{C\,D\,(C+D+2)(C+D+3)} - 1\right]$$

$$\text{Mode} = \frac{A\,(D-1) + B\,(C-1)}{C+D-2}\,,\ \text{unless } C = D = 1$$

Median, Q1, Q3, qMean, qMode: no simple closed form

Notes

1. Although C and D have no upper bound, in fact, they seldom exceed 10. If optimum values are much greater than this, the response will often be nearly flat.
2. If both C and D are large, the distribution is roughly symmetrical and some other model is indicated.
3. The beta distribution is often used to mimic other distributions. When suitably transformed and normalized, a vector of random variables can almost always be modeled as Beta.

Aliases and Special Cases

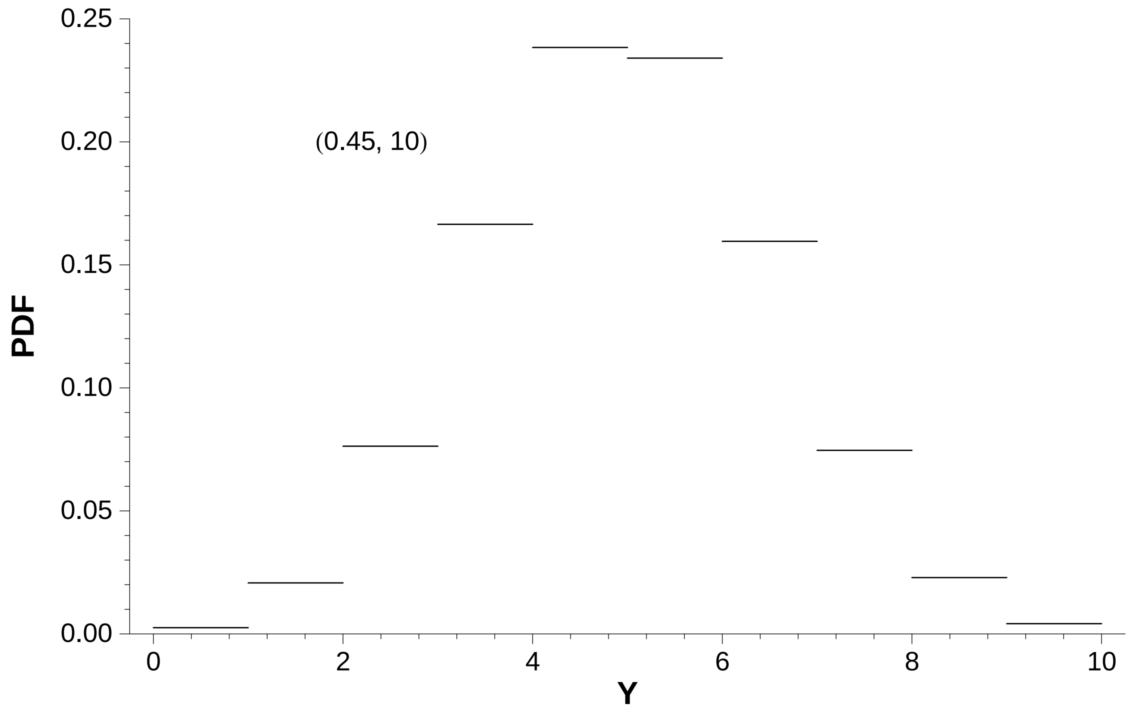
1. Beta(0, 1, C, 1) is often called the *Power-function* distribution.

Characterizations

1. If X_j^2, j = 1, 2 ~ standard *Chi-square* with ν_j degrees of freedom, respectively, then $Z = \frac{X_1^2}{X_1^2 + X_2^2}$ ~Beta(0, 1, $\nu_1/2$, $\nu_2/2$).
2. More generally, $Z = \frac{W_1}{W_1 + W_2}$ ~Beta(0, 1, p_1, p_2) if W_j ~**Gamma**(0, σ, p_j), for any scale (σ).
3. If $Z_1, Z_2, \ldots, Z_N$ ~**Uniform**(0, 1) are sorted to give the corresponding order statistics $Z'_1 \le Z'_2 \le \ldots \le Z'_N$, then the s[th]-order statistic Z'_s ~Beta(0, 1, s, N – s + 1).

Binomial(A,B)

$y = 0, 1, 2, \ldots, \quad 0 < A < 1, \quad y, 2 \le B$



$$\text{PDF} = \binom{B}{y} A^{y} (1 - A)^{B - y}$$

Parameters -- A (p): Prob(success), B (N): Number of Bernoulli trials (constant)

Moments, etc.

$$\text{Mean} = A\,B$$

$$\text{Variance} = A\,(1 - A)\,B$$

$$\text{Mode} = \text{int}\left(A\,(B + 1)\right)$$

Notes

1. In the literature, B may be any positive integer.
2. If A (B + 1) is an integer, Mode also equals A (B + 1) – 1.
3. *Regress+* requires B to be Constant.

Aliases and Special Cases

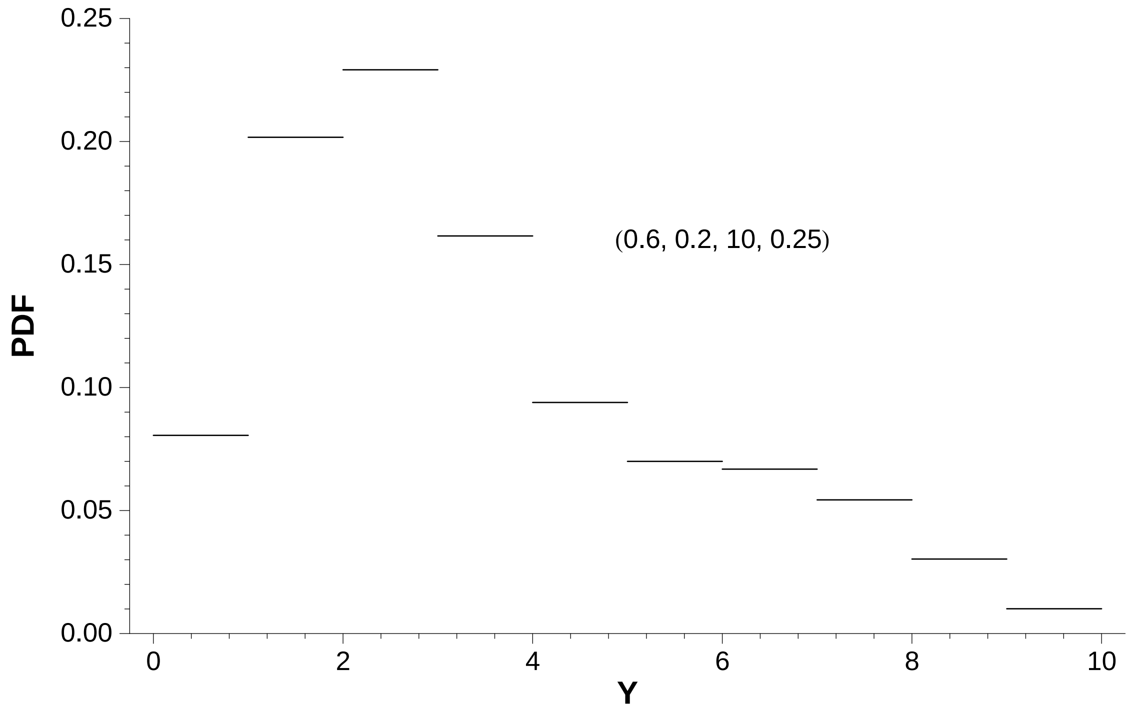
1. Although disallowed here because of incompatibility with several *Regress+* features, Binomial(A,1) is called the *Bernoulli* distribution.

Characterizations

1. The probability of exactly y successes, each having Prob(success) = A, in a series of B independent trials is ~Binomial(A, B).

Binomial(A,C)&Binomial(B,C)

y = 0, 1, 2, …, $0 < B < A < 1$, $y, 3 \le C$, $0 < p < 1$



$$\text{PDF} = \binom{C}{y}\left[p\,A^{y}\left(1-A\right)^{C-y} + \left(1-p\right)B^{y}\left(1-B\right)^{C-y}\right]$$

Parameters -- A, B (π_1, π_2): Prob(success), C (N): Number of Bernoulli trials (constant), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = C\left[p\,A + \left(1-p\right)B\right]$$

$$\text{Variance} = C\left[p\,A\left(1-A\right) + \left(1-p\right)\left(p\,C\left(A-B\right)^{2} + B\left(1-B\right)\right)\right]$$

Mode: no simple closed form

Notes

1. Here, parameter A is stipulated to be the Component with the larger Prob(success).
2. Parameter C **must** be at least 3 in order for this distribution to be *identifiable*, i.e., well-defined.
3. *Regress+* requires C to be Constant.
4. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
5. **Warning!** Mixtures usually have several local optima.

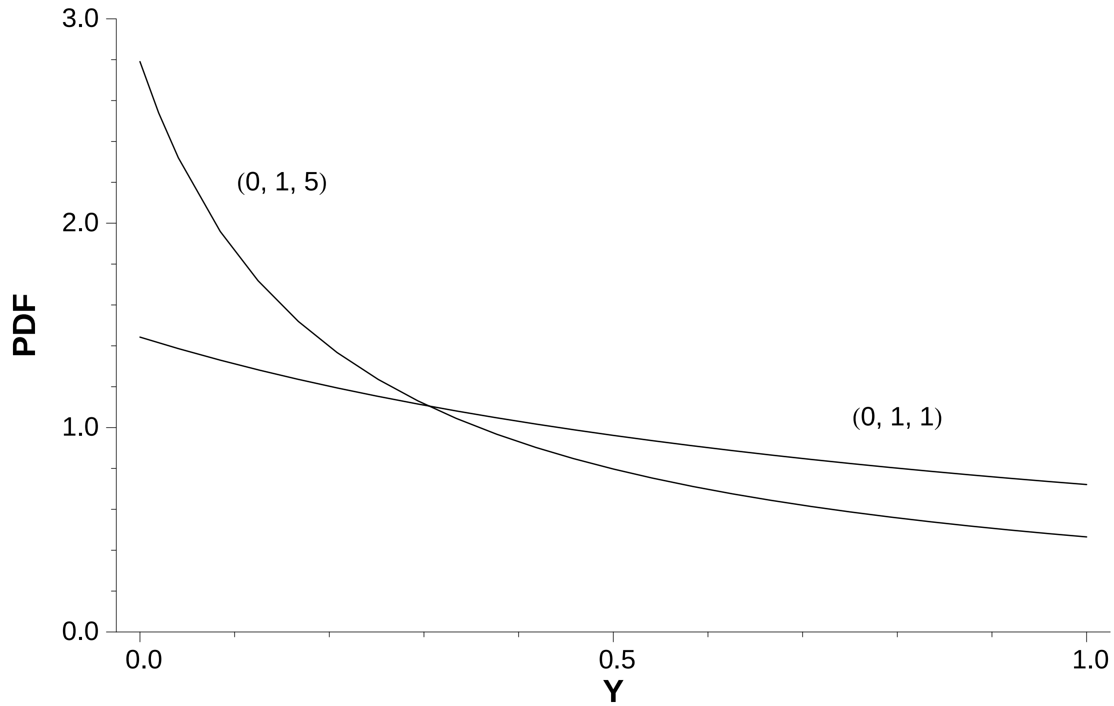
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Bradford(A,B,C)

$A < y < B, \quad C > 0$



$$\text{PDF} = \frac{C}{\left[C\left(y - A\right) + B - A\right]\log\left(C + 1\right)}$$

$$\text{CDF} = \frac{\log\left(1 + \frac{C\left(y - A\right)}{B - A}\right)}{\log\left(C + 1\right)}$$

Parameters -- A: Location, B: Scale (upper bound), C: Shape

Moments, etc. $\left[k \equiv \log\left(C + 1\right)\right]$

$$\text{Mean} = \frac{C\left(B - A\right) + k\left[A\left(C + 1\right) - B\right]}{C\,k}$$

$$\text{Variance} = \frac{\left(B - A\right)^2\left[C\left(k - 2\right) + 2\,k\right]}{2\,C\,k^2}$$

$$\text{Skewness} = \frac{\sqrt{2}\left[12\,C^2 - 9\,k\,C\,(C+2) + 2\,k^2\left(C\,(C+3)+3\right)\right]}{\sqrt{C\left[C\,(k-2)+2\,k\right]}\left[3\,C\,(k-2)+6\,k\right]}$$

$$\text{Kurtosis} = \frac{C^3\,(k-3)\left[k\,(3\,k-16)+24\right] + 12\,k\,C^2\,(k-4)\,(k-3) + 6\,C\,k^2\,(3\,k - 14) + 12\,k^3}{3\,C\left[C\,(k-2)+2\,k\right]^2}$$

$$\text{Mode} = A$$

$$\text{Median} = \frac{1}{C}\left[A\,(C+1) - B + (B-A)\,\sqrt{C+1}\right]$$

$$Q1 = \frac{1}{C}\left[A\,(C+1) - B + (B-A)\,\sqrt[4]{C+1}\right] \qquad Q3 = \frac{1}{C}\left[A\,(C+1) - B + (B-A)\,\sqrt[4]{(C+1)^3}\right]$$

$$\text{qMean} = \frac{\log\left(\dfrac{C}{\log\,(C+1)}\right)}{\log\,(C+1)} \qquad \text{qMode} = 0$$

$$\text{RandVar} = \frac{1}{C}\left[A\,(C+1) - B + (B-A)\,(C+1)^u\right]$$

Notes

1. With the log-likelihood criterion, parameter C is often flat.

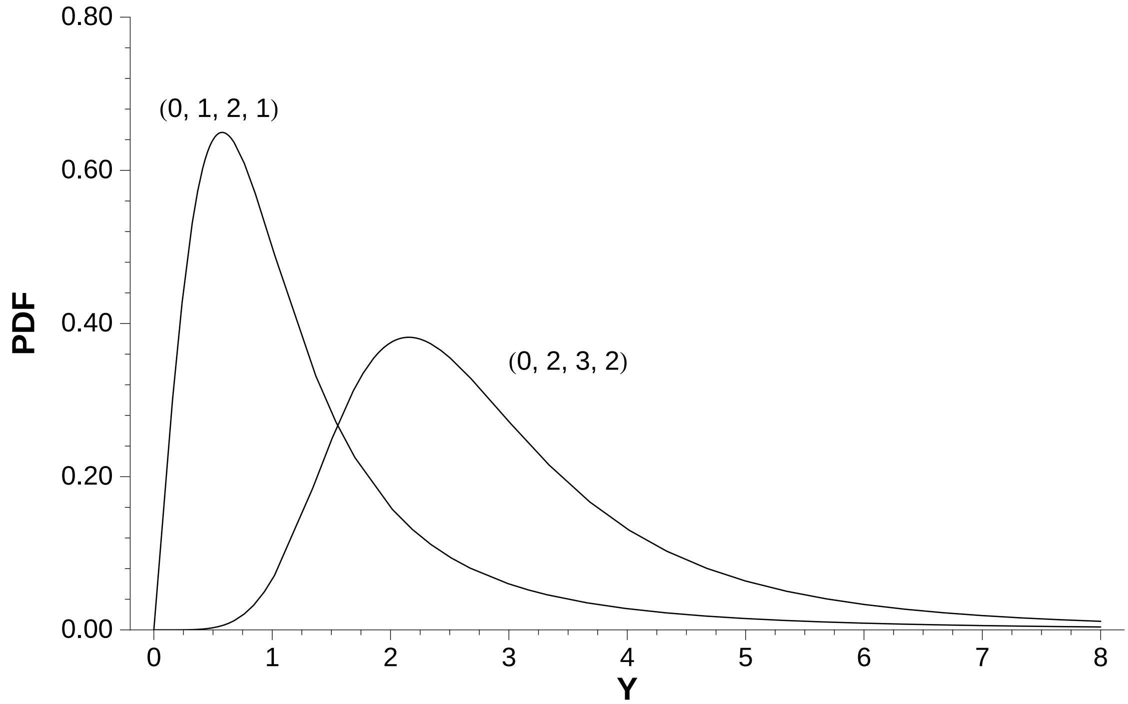
Aliases and Special Cases

Characterizations

1. The Bradford distribution has been used to model the distribution of references among several sources.

Burr(A,B,C,D)

$y > A, \quad B > 0, \quad 0 < C, D \le 100$



$$\text{PDF} = \frac{C\,D}{B}\left(\frac{y-A}{B}\right)^{-C-1}\left(1+\left(\frac{y-A}{B}\right)^{-C}\right)^{-D-1}$$

$$\text{CDF} = \left(1+\left(\frac{y-A}{B}\right)^{-C}\right)^{-D}$$

Parameters -- A: Location, B: Scale, C, D: Shape

Moments, etc. $\left[k \equiv \Gamma(D)\,\Gamma\left(1-\frac{2}{C}\right)\Gamma\left(\frac{2}{C}+D\right)-\Gamma^2\left(1-\frac{1}{C}\right)\Gamma^2\left(\frac{1}{C}+D\right)\right]$

$$\text{Mean} = A + \frac{B\,\Gamma\left(1-\frac{1}{C}\right)\Gamma\left(\frac{1}{C}+D\right)}{\Gamma(D)}$$

$$\text{Variance} = \frac{k\,B^2}{\Gamma^2(D)}$$

$$\text{Skewness} = \frac{\Gamma^3(D)}{\sqrt{k^3}}\left[\frac{2\,\Gamma^3\left(1-\frac{1}{C}\right)\Gamma^3\left(\frac{1}{C}+D\right)}{\Gamma^3(D)} - \frac{3\,\Gamma\left(1-\frac{2}{C}\right)\Gamma\left(1-\frac{1}{C}\right)\Gamma\left(\frac{1}{C}+D\right)\Gamma\left(\frac{2}{C}+D\right)}{\Gamma^2(D)} + \frac{\Gamma\left(1-\frac{3}{C}\right)\Gamma\left(\frac{3}{C}+D\right)}{\Gamma(D)}\right]$$

Kurtosis =

$$-3 + \frac{\Gamma^4(D)}{k^2}\left[\frac{-3\,\Gamma^4\left(1-\frac{1}{C}\right)\Gamma^4\left(\frac{1}{C}+D\right)}{\Gamma^4(D)} + \frac{6\,\Gamma\left(1-\frac{2}{C}\right)\Gamma^2\left(1-\frac{1}{C}\right)\Gamma^2\left(\frac{1}{C}+D\right)\Gamma\left(\frac{2}{C}+D\right)}{\Gamma^3(D)}\right] -$$

$$\frac{\Gamma^4(D)}{k^2}\left[\frac{4\,\Gamma\left(1-\frac{3}{C}\right)\Gamma\left(1-\frac{1}{C}\right)\Gamma\left(\frac{1}{C}+D\right)\Gamma\left(\frac{3}{C}+D\right)}{\Gamma^2(D)} - \frac{\Gamma\left(1-\frac{4}{C}\right)\Gamma\left(\frac{4}{C}+D\right)}{\Gamma(D)}\right]$$

$$\text{Mode} = A + B\sqrt[C]{\frac{C\,D-1}{C+1}}\ ,\ \text{iff } C\,D > 1, \text{ else Mode} = A \text{ (and qMode} = 0)$$

$$\text{Median} = A + B\left(\sqrt[D]{2}-1\right)^{-\frac{1}{C}}$$

$$\text{Q1} = A + B\left(\sqrt[D]{4}-1\right)^{-\frac{1}{C}} \qquad \text{Q3} = A + B\left(\sqrt[D]{\frac{4}{3}}-1\right)^{-\frac{1}{C}}$$

$$\text{qMean} = \left[1 + \Gamma^{C}(D)\,\Gamma^{-C}\left(1-\frac{1}{C}\right)\Gamma^{-C}\left(\frac{1}{C}+D\right)\right]^{-D} \qquad \text{qMode} = \left(1 + \frac{C+1}{C\,D-1}\right)^{-D}$$

$$\text{RandVar} = A + B\left(\frac{1}{\sqrt[D]{u}}-1\right)^{-\frac{1}{C}}$$

Notes

1. With the log-likelihood criterion, parameter C is often flat.

Aliases and Special Cases

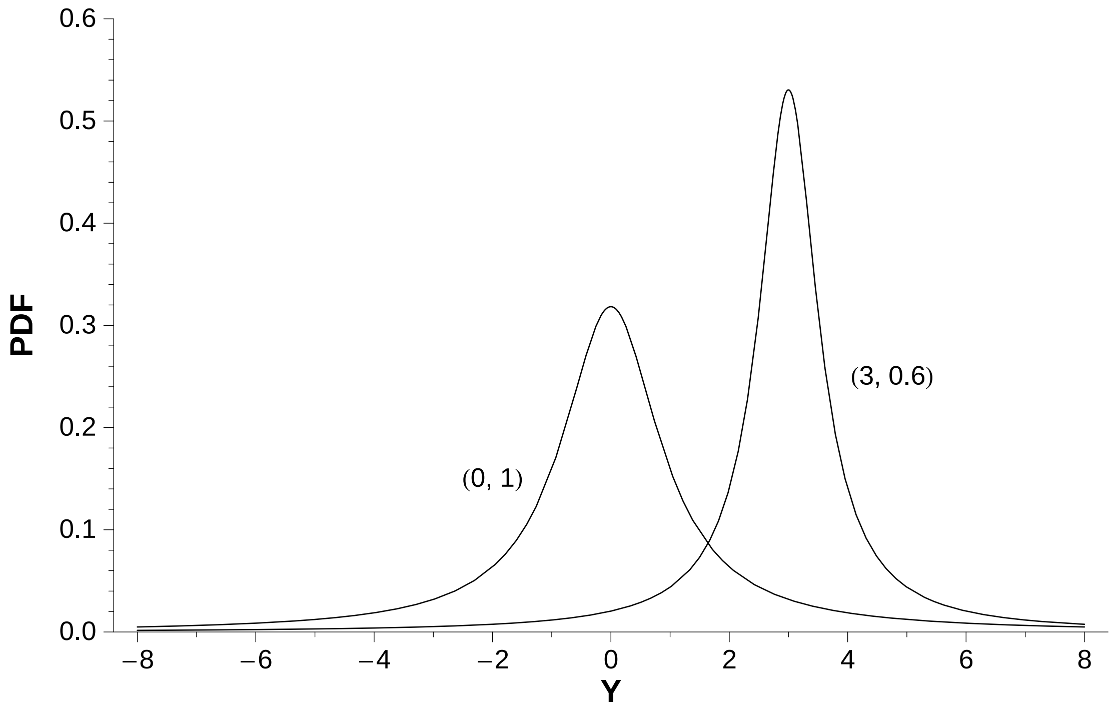
1. The Burr distribution, with D = 1, is often called the **Fisk** or *LogLogistic* distribution.

Characterizations

1. The Burr distribution is a generalization of the **Fisk** distribution.

Cauchy(A,B)

B > 0



$$\text{PDF} = \frac{1}{\pi B \left[1 + \left(\frac{y - A}{B} \right)^2 \right]}$$

$$\text{CDF} = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{y - A}{B} \right)$$

Parameters -- A (θ): Location, B (λ): Scale

Moments, etc.

This distribution has no finite moments because the corresponding integrals do not converge.

$$\text{Median} = \text{Mode} = A$$

$$Q1 = A - B \qquad Q3 = A + B$$

$$\text{qMode} = 0.5$$

$$\text{RandVar} = A + B \tan\left[\pi\left(u - \frac{1}{2}\right)\right]$$

Notes

1. Since there are no finite moments, the location parameter (ostensibly the mean) does not have its usual interpretation for a symmetrical distribution.

Aliases and Special Cases

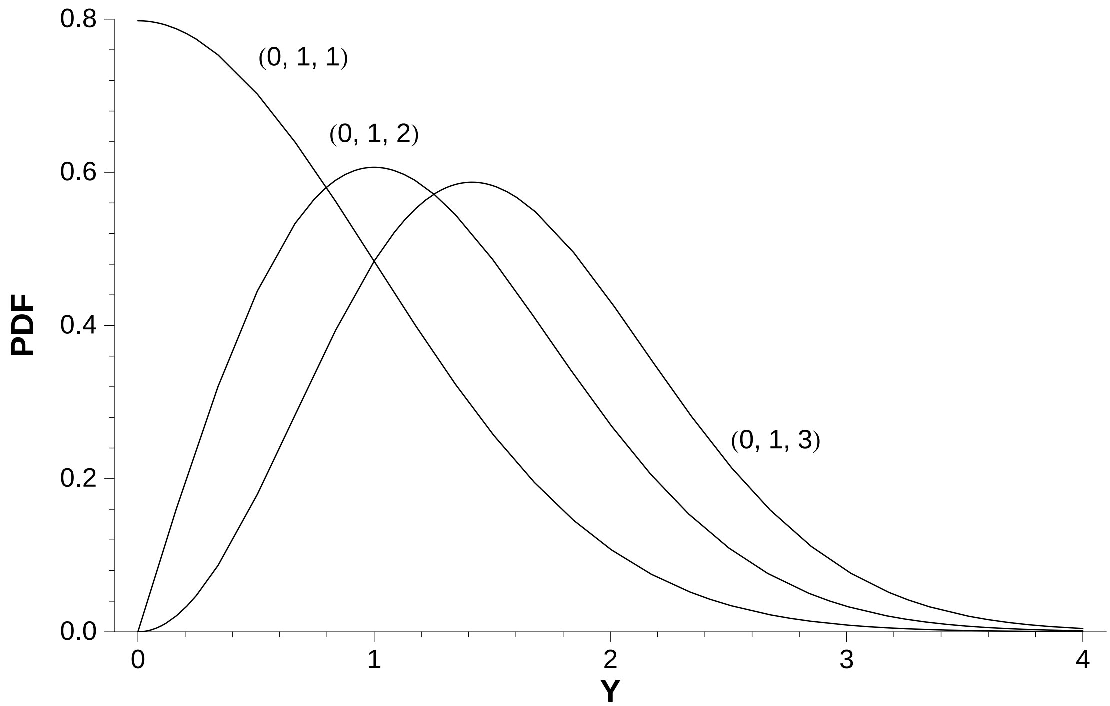
1. The Cauchy distribution is sometimes called the *Lorentz* distribution.

Characterizations

1. If U and V are ~**Normal**(0, 1), the ratio U/V ~Cauchy(0, 1).
2. If Z ~Cauchy, then $W = (a + b*Z)^{-1}$ ~Cauchy.
3. If particles emanate from a fixed point, their points of impact on a straight line ~Cauchy.

Chi(A,B,C)

$y > A, \quad B > 0, \quad 0 < C \leq 100$



$$\text{PDF} = \frac{\left(\frac{y-A}{B}\right)^{C-1} \exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right)}{2^{\frac{C}{2}-1} B\, \Gamma\left(\frac{C}{2}\right)}$$

$$\text{CDF} = \Gamma\left(\frac{C}{2}, \frac{1}{2}\left(\frac{y-A}{B}\right)^2\right)$$

Parameters -- A: Location, B: Scale, C (ν): Shape (also, degrees of freedom)

Moments, etc.

$$\text{Mean} = A + \frac{\sqrt{2}\, B\, \Gamma\left(\frac{C+1}{2}\right)}{\Gamma\left(\frac{C}{2}\right)}$$

$$\text{Variance} = B^2 \left[C - \frac{2\,\Gamma^2\left(\frac{C+1}{2}\right)}{\Gamma^2\left(\frac{C}{2}\right)}\right]$$

$$\text{Skewness} = \frac{\sqrt{2}\left[4\,\Gamma^3\left(\frac{C+1}{2}\right) + \Gamma^2\left(\frac{C}{2}\right)\left(2\,\Gamma\left(\frac{C+3}{2}\right) - 3\,C\,\Gamma\left(\frac{C+1}{2}\right)\right)\right]}{\Gamma^3\left(\frac{C}{2}\right)\left[C - \frac{2\,\Gamma^2\left(\frac{C+1}{2}\right)}{\Gamma^2\left(\frac{C}{2}\right)}\right]^{\frac{3}{2}}}$$

$$\text{Kurtosis} = \frac{2\,C\left(1-C\right)\Gamma^4\left(\frac{C}{2}\right) - 24\,\Gamma^4\left(\frac{C+1}{2}\right) + 8\left(2\,C-1\right)\Gamma^2\left(\frac{C}{2}\right)\Gamma^2\left(\frac{C+1}{2}\right)}{\left[C\,\Gamma^2\left(\frac{C}{2}\right) - 2\,\Gamma^2\left(\frac{C+1}{2}\right)\right]^2}$$

$$\text{Mode} = A + B\sqrt{C-1}$$

Median, Q1, Q3, qMean, qMode: no simple closed form

Notes

1. In the literature, $C > 0$. The restrictions shown above are required for convergence when the data are left-skewed and to ensure the existence of a Mode.

Aliases and Special Cases

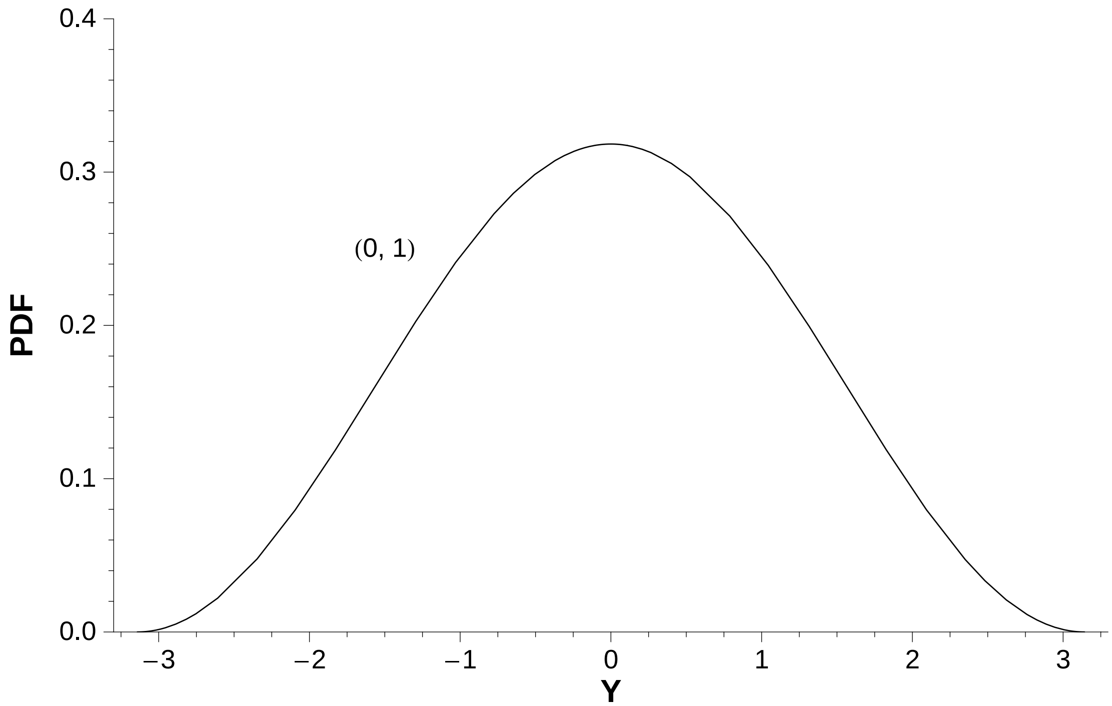
1. Chi(A, B, 1) is the **HalfNormal** distribution.
2. Chi(0, B, 2) is the *Rayleigh* distribution.
3. Chi(0, B, 3) is the *Maxwell* distribution.

Characterizations

1. If Z ~Chi-square, its positive square root is ~Chi.
2. If X, Y ~**Normal**(0, B), the distance from the origin to the point (X, Y) is ~Rayleigh(B).
3. If a spatial pattern is generated by a Poisson process, the distance between any pattern element and its nearest neighbor is ~Rayleigh.
4. The speed of a random molecule, at any temperature, is ~Maxwell.

Cosine(A,B)

$A - \pi B \le y \le A + \pi B, \quad B > 0$



$$\text{PDF} = \frac{1}{2\pi B}\left[1 + \cos\left(\frac{y-A}{B}\right)\right]$$

$$\text{CDF} = \frac{1}{2\pi}\left[\pi + \frac{y-A}{B} + \sin\left(\frac{y-A}{B}\right)\right]$$

Parameters -- A: Location, B: Scale

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = \left(\frac{\pi^2}{3} - 2\right) B^2$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = \frac{-6\left(\pi^4 - 90\right)}{5\left(\pi^2 - 6\right)^2} \approx -0.5938$$

$$Q1 \approx A - 0.8317\,B \qquad Q3 \approx A + 0.8317\,B$$

$$qMean = qMode = 0.5$$

Notes

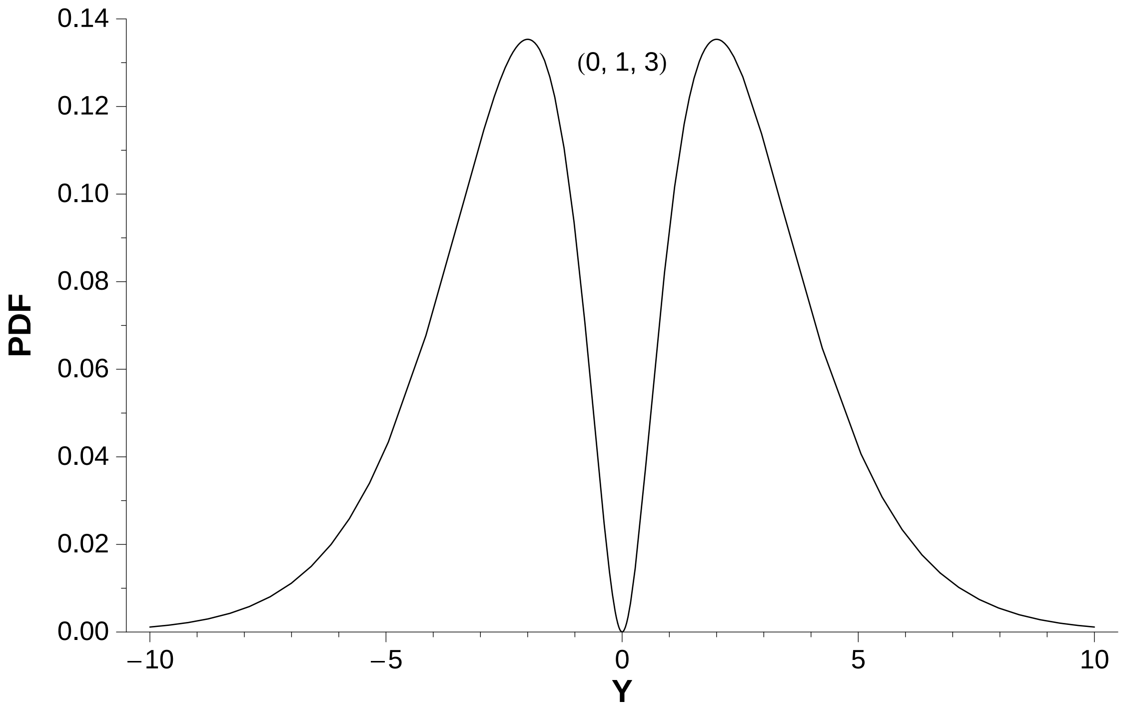
Aliases and Special Cases

Characterizations

1. The Cosine distribution is sometimes used as a simple, and more computationally tractable, approximation to the **Normal** distribution.

DoubleGamma(A,B,C)

B, C > 0



$$\text{PDF} = \frac{1}{2\,B\,\Gamma(C)} \left| \frac{y-A}{B} \right|^{C-1} \exp\left(-\left| \frac{y-A}{B} \right| \right)$$

$$\text{CDF} = \begin{cases} \frac{1}{2} - \frac{1}{2}\,\Gamma\left(C, \left| \frac{y-A}{B} \right| \right), & y \le A \\ \frac{1}{2} + \frac{1}{2}\,\Gamma\left(C, \left| \frac{y-A}{B} \right| \right), & y > A \end{cases}$$

Parameters -- A: Location, B: Scale, C: Shape

Moments, etc.

$$\text{Mean} = \text{Median} = A$$

$$\text{Variance} = C\,(C+1)\,B^2$$

$$\text{Skewness} = 0$$

Kurtosis, Mode: not applicable (bimodal)

Q1, Q3: no simple closed form

qMean = 0.5

RandVar = RandGamma , with a random sign

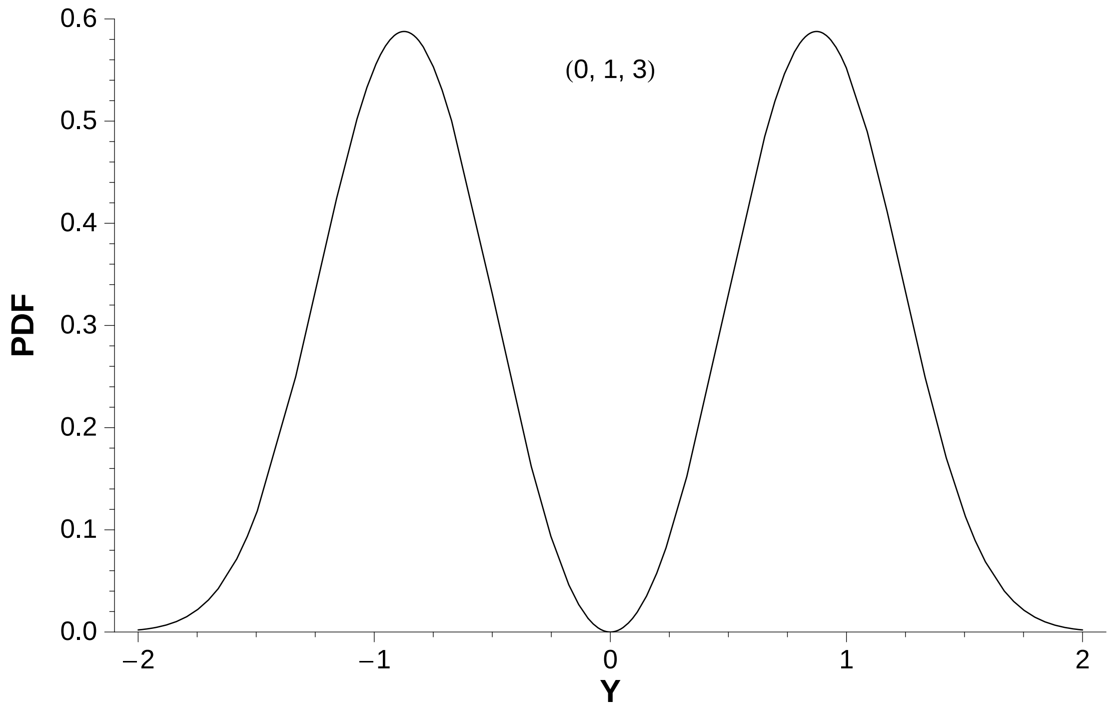
Notes

Aliases and Special Cases

Characterizations

1. The DoubleGamma distribution is the signed version of the **Gamma** distribution.

DoubleWeibull(A,B,C) B, C > 0



$$\text{PDF} = \frac{C}{2B}\left|\frac{y-A}{B}\right|^{C-1} \exp\left(-\left|\frac{y-A}{B}\right|^{C}\right)$$

$$\text{CDF} = \begin{cases} \frac{1}{2}\exp\left(-\left|\frac{y-A}{B}\right|^{C}\right), & y \le A \\ 1-\frac{1}{2}\exp\left(-\left|\frac{y-A}{B}\right|^{C}\right), & y > A \end{cases}$$

Parameters -- A: Location, B: Scale, C: Shape

Moments, etc.

$$\text{Mean} = \text{Median} = A$$

$$\text{Variance} = \Gamma\left(\frac{C+2}{C}\right) B^2$$

$$\text{Skewness} = 0$$

Kurtosis, Mode: not applicable (bimodal)

Q1, Q3: no simple closed form

qMean = 0.5

RandVar = RandWeibull , with a random sign

Notes

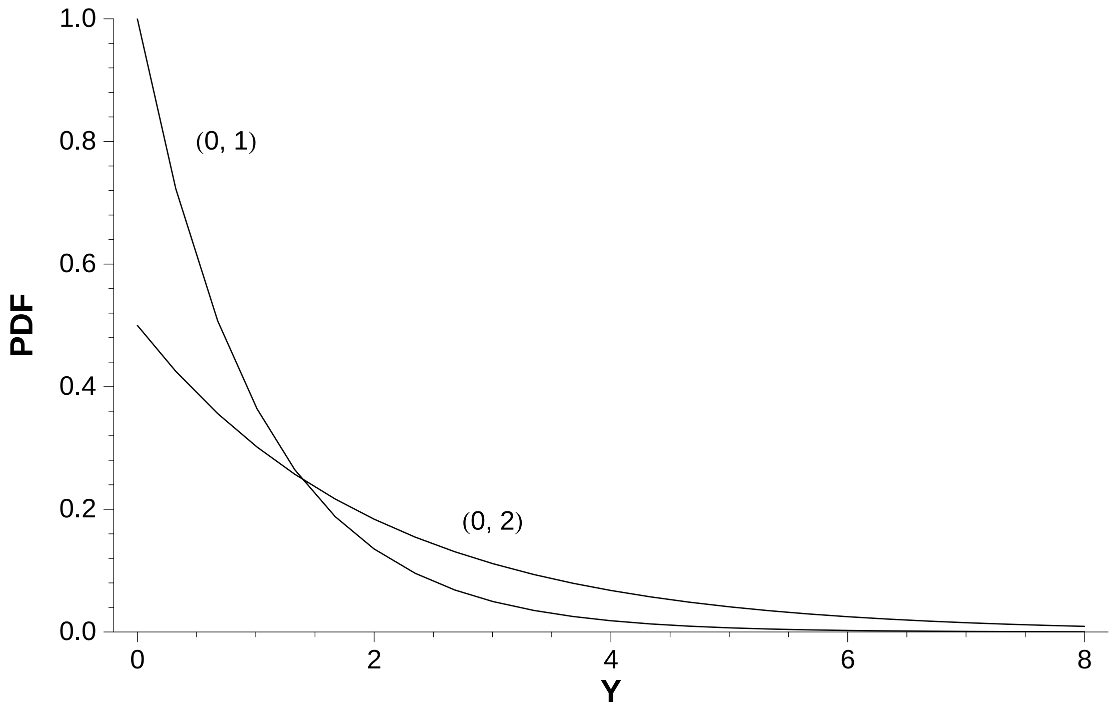
Aliases and Special Cases

Characterizations

1. The DoubleWeibull distribution is the signed version of the **Weibull** distribution.

Exponential(A,B)

$y \geq A, \quad B > 0$



$$\text{PDF} = \frac{1}{B} \exp\left(\frac{A - y}{B}\right)$$

$$\text{CDF} = 1 - \exp\left(\frac{A - y}{B}\right)$$

Parameters -- A (θ): Location, B (λ): Scale

Moments, etc.

$$\text{Mean} = A + B$$

$$\text{Variance} = B^2$$

$$\text{Skewness} = 2$$

$$\text{Kurtosis} = 6$$

$$\text{Mode} = A$$

$$\text{Median} = A + B \log(2)$$

$$Q1 = A + B\log\left(\frac{4}{3}\right) \qquad Q3 = A + B\log(4)$$

$$qMean = \frac{e-1}{e} \approx 0.6321 \qquad qMode = 0$$

$$RandVar = A - B\log(u)$$

Notes

1. The one-parameter version of this distribution, Exponential(0,B), is far more common than the more general formulation shown here.

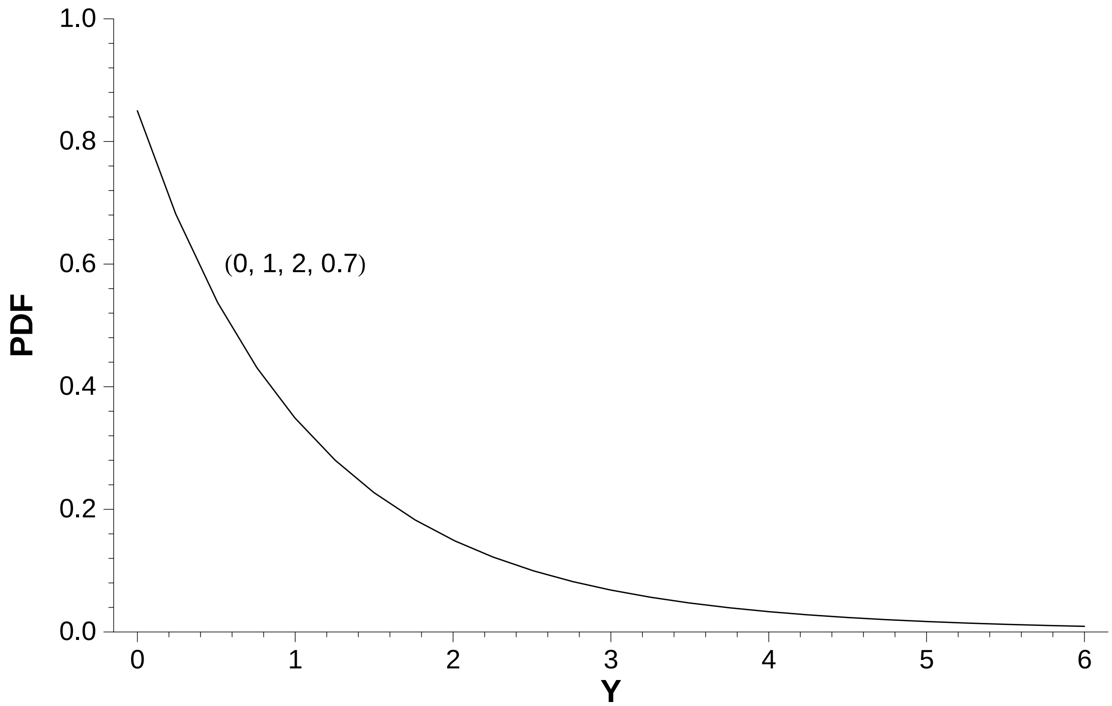
Aliases and Special Cases

1. The Exponential distribution is sometimes called the Negative exponential distribution.
2. The discrete version of the Exponential distribution is the **Geometric** distribution.

Characterizations

1. If the future lifetime of a system at any time, t, has the same distribution for all t, then this distribution is the Exponential distribution. This is known as the memoryless property.

Expo(A,B)&Expo(A,C) $y \geq A, \quad B, C > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{B} \exp\left(\frac{A-y}{B}\right) + \frac{(1-p)}{C} \exp\left(\frac{A-y}{C}\right)$$

$$\text{CDF} = p \text{ stdExponentialCDF}\left(\frac{y-A}{B}\right) + (1-p) \text{ stdExponentialCDF}\left(\frac{y-A}{C}\right)$$

Parameters -- A (θ): Location, B, C (λ_1, λ_2): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = A + p\,B + (1-p)\,C$$

$$\text{Variance} = C^2 + 2\,B\,(B-C)\,p - (B-C)^2\,p^2$$

$$\text{Mode} = A$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

Aliases and Special Cases

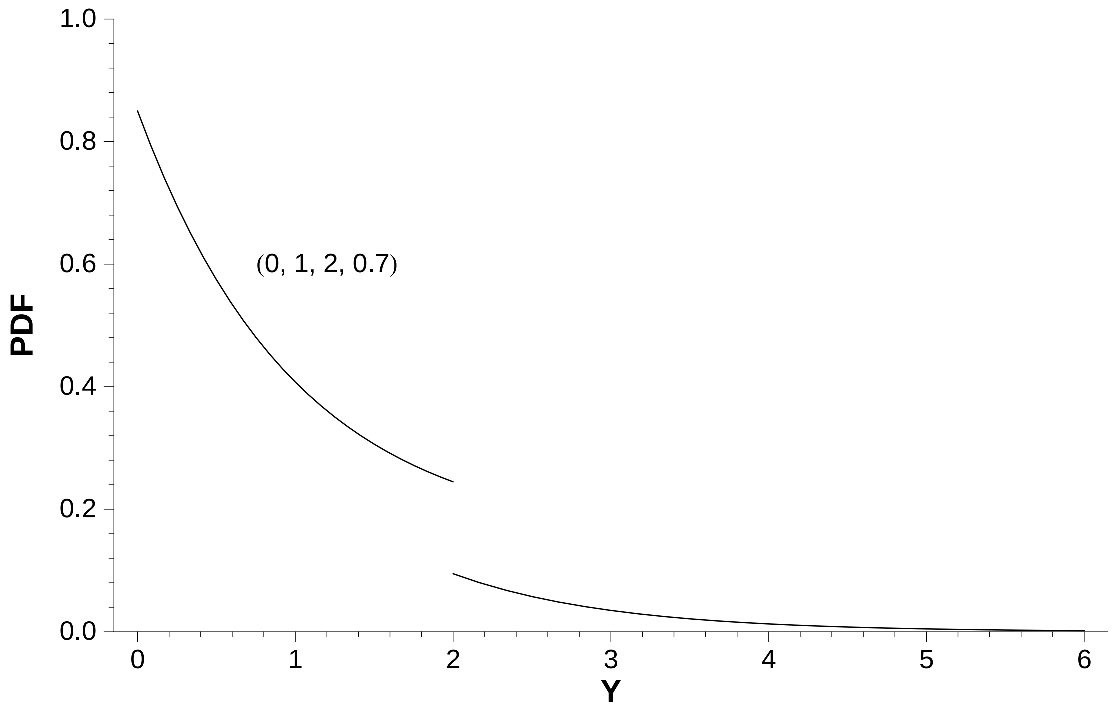
1. This mixture, when applied to traffic analysis, is often called the *Schuhl* distribution.

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Expo(A,B)&Uniform(A,C)

$A \le y < C, \quad B > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{B}\exp\left(\frac{A-y}{B}\right) + \frac{(1-p)\,(y<C)}{C-A}$$

$$\text{CDF} = \begin{cases} p\ \text{stdExponentialCDF}\left(\frac{y-A}{B}\right) + (1-p)\left(\frac{y-A}{C-A}\right), & y < C \\ p\ \text{stdExponentialCDF}\left(\frac{y-A}{B}\right) + (1-p), & y \ge C \end{cases}$$

Parameters -- A (θ): Location, B (λ), C : Scale (C = upper bound of Uniform(A,C)), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \frac{A + C + p\,(A + 2\,B - C)}{2}$$

$$\text{Variance} = p\left[B^2 + (A+B)^2\right] - \frac{(p-1)\left(A^2 + A\,C + C^2\right)}{3} - \frac{\left[A + C + p\,(A + 2\,B - C)\right]^2}{4}$$

$$\text{Mode} = A$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

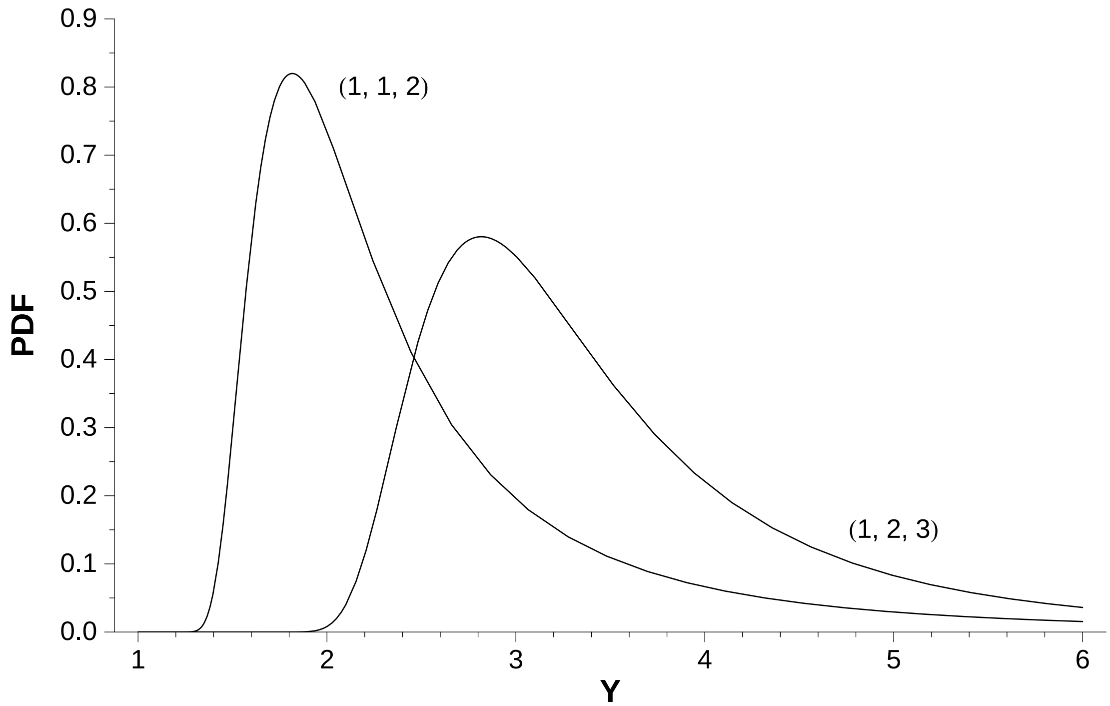
1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

ExtremeLB(A,B,C) $y > A, \quad B > 0, \quad 0 < C \leq 100$



$$\text{PDF} = \frac{C}{B}\left(\frac{y-A}{B}\right)^{-C-1}\exp\left(-\left(\frac{y-A}{B}\right)^{-C}\right)$$

$$\text{CDF} = \exp\left(-\left(\frac{y-A}{B}\right)^{-C}\right)$$

Parameters -- A (ξ): Location, B (θ): Scale, C (k): Shape

Moments, etc. (see Note #3.)

$$\text{Mean} = A + B\,\Gamma\left(\frac{C-1}{C}\right)$$

$$\text{Variance} = B^2\left[\Gamma\left(\frac{C-2}{C}\right) - \Gamma^2\left(\frac{C-1}{C}\right)\right]$$

$$\text{Skewness} = \frac{\Gamma\left(\frac{C-3}{C}\right) - 3\,\Gamma\left(\frac{C-2}{C}\right)\Gamma\left(\frac{C-1}{C}\right) + 2\,\Gamma^3\left(\frac{C-1}{C}\right)}{\sqrt{\left[\Gamma\left(\frac{C-2}{C}\right) - \Gamma^2\left(\frac{C-1}{C}\right)\right]^3}}$$

$$\text{Kurtosis} = -6 + \frac{\Gamma\left(\frac{C-4}{C}\right) - 4\,\Gamma\left(\frac{C-3}{C}\right)\Gamma\left(\frac{C-1}{C}\right) + 3\,\Gamma^2\left(\frac{C-2}{C}\right)}{\left[\Gamma\left(\frac{C-2}{C}\right) - \Gamma^2\left(\frac{C-1}{C}\right)\right]^2}$$

$$\text{Mode} = A + B\,\sqrt[C]{\frac{C}{1+C}}$$

$$\text{Median} = A + \frac{B}{\sqrt[C]{\log(2)}}$$

$$\text{Q1} = A + \frac{B}{\sqrt[C]{\log(4)}} \qquad \text{Q3} = A + \frac{B}{\sqrt[C]{\log\left(\frac{4}{3}\right)}}$$

$$\text{qMean} = \exp\left(-\Gamma^{-C}\left(\frac{C-1}{C}\right)\right) \qquad \text{qMode} = \exp\left(-\frac{C+1}{C}\right)$$

$$\text{RandVar} = A + B\left(-\log(u)\right)^{-\frac{1}{C}}$$

Notes

1. The name ExtremeLB does not appear in the literature. It was chosen here simply to indicate one type of extreme-value distribution with a lower bound.
2. In the literature, C > 0. The restriction shown above is required for convergence when the data are left-skewed.
3. Moment k exists if C > k.

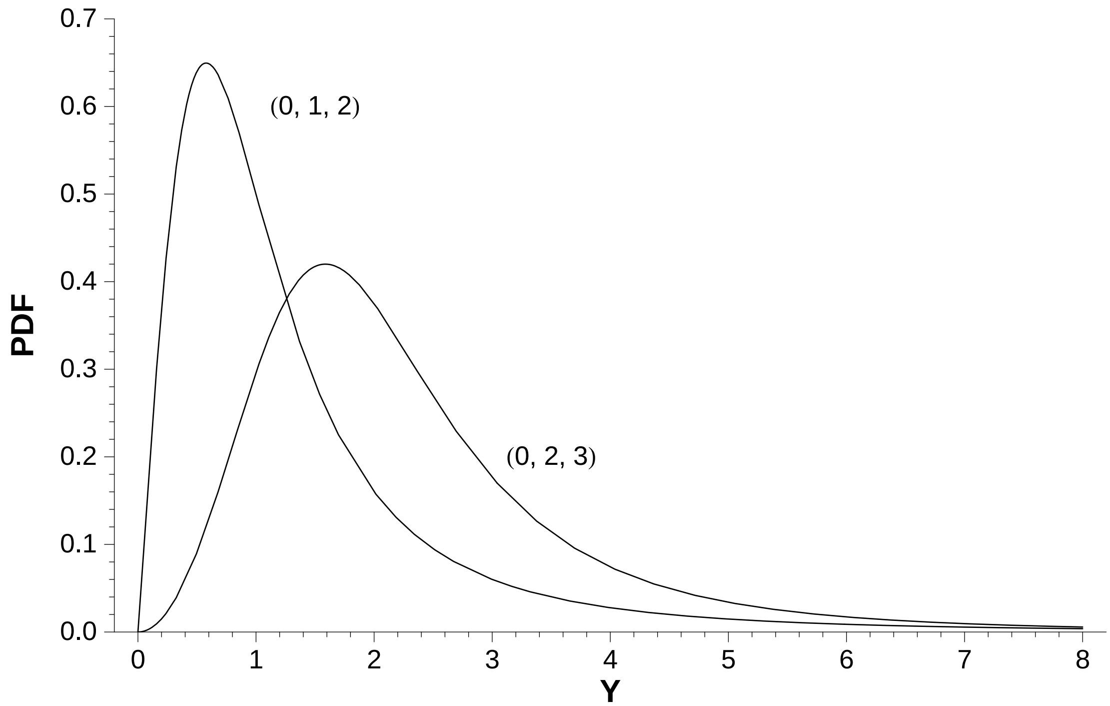
Aliases and Special Cases

1. The corresponding distribution with an upper bound is **Weibull**(–y).

Characterizations

1. Extreme-value distributions are the limiting distributions, as N --> infinity, of the greatest value among N i.i.d. variates selected from a continuous distribution. By replacing y with –y, the smallest values may be modeled.

Fisk(A,B,C) $y > A, \quad B > 0, \quad 0 < C \leq 100$



$$\text{PDF} = \frac{C}{B} \frac{\left(\frac{y-A}{B}\right)^{C-1}}{\left[1 + \left(\frac{y-A}{B}\right)^{C}\right]^2}$$

$$\text{CDF} = \frac{1}{1 + \left(\frac{y-A}{B}\right)^{-C}}$$

Parameters -- A: Location, B: Scale, C: Shape

Moments, etc. (see Note #3.)

$$\text{Mean} = A + B\left[\frac{\pi}{C}\csc\left(\frac{\pi}{C}\right)\right]$$

$$\text{Variance} = B^2\left[\frac{2\,\pi}{C}\csc\left(\frac{2\,\pi}{C}\right) - \left[\frac{\pi}{C}\csc\left(\frac{\pi}{C}\right)\right]^2\right]$$

$$\text{Skewness} = \frac{2\pi^2 \csc^3\left(\frac{\pi}{C}\right) - 6 C \pi \csc\left(\frac{\pi}{C}\right) \csc\left(\frac{2\pi}{C}\right) + 3 C^2 \csc\left(\frac{3\pi}{C}\right)}{\sqrt{\pi}\left[2 C \csc\left(\frac{2\pi}{C}\right) - \pi \csc^2\left(\frac{\pi}{C}\right)\right]^{\frac{3}{2}}}$$

$$\text{Kurtosis} = \frac{-3\pi^3 \csc^4\left(\frac{\pi}{C}\right) - 12 C^2 \pi \csc\left(\frac{\pi}{C}\right) \csc\left(\frac{3\pi}{C}\right) + 4 C^3 \csc\left(\frac{4\pi}{C}\right) + 6 C \pi^2 \csc^3\left(\frac{\pi}{C}\right) \sec\left(\frac{\pi}{C}\right)}{\pi\left[\pi \csc^2\left(\frac{\pi}{C}\right) - 2 C \csc\left(\frac{2\pi}{C}\right)\right]^2} - 3$$

$$\text{Mode} = A + B \sqrt[C]{\frac{C-1}{C+1}}$$

$$\text{Median} = A + B$$

$$\text{Q1} = A + \frac{B}{\sqrt[C]{3}} \qquad \text{Q3} = A + B \sqrt[C]{3}$$

$$\text{qMean} = \frac{1}{1 + \left[\frac{\pi}{C} \csc\left(\frac{\pi}{C}\right)\right]^{-C}} \qquad \text{qMode} = \frac{C-1}{2C}$$

$$\text{RandVar} = A + B \sqrt[C]{\frac{u}{1-u}}$$

Notes

1. The Fisk distribution is right-skewed.
2. To model a left-skewed distribution, try modeling $w = -y$.
3. Moment k exists if $C > k$.

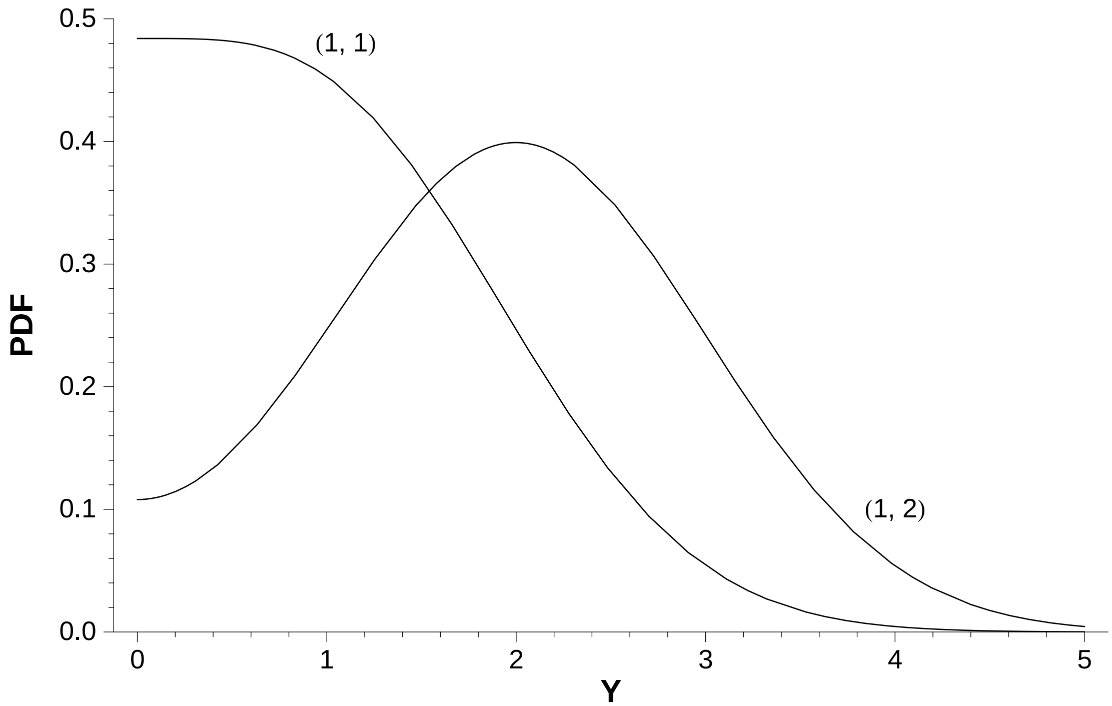
Aliases and Special Cases

1. The Fisk distribution is also known as the *LogLogistic* distribution.

Characterizations

1. The Fisk distribution is often used in income and lifetime analysis.

FoldedNormal(A,B) $y \geq 0, \quad A \geq 0, \quad B > 0$



$$\text{PDF} = \frac{1}{B}\sqrt{\frac{2}{\pi}}\cosh\left(\frac{A\,y}{B^2}\right)\exp\left(-\frac{1}{2}\,\frac{y^2 + A^2}{B^2}\right)$$

$$\text{CDF} = \Phi\left(\frac{y - A}{B}\right) - \Phi\left(\frac{-y - A}{B}\right)$$

Parameters -- A (μ): Location, B (σ): Scale, both for the corresponding unfolded **Normal**

Moments, etc.

$$\text{Mean} = B\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2}\left(\frac{A}{B}\right)^2\right) - A\left[1 - 2\,\Phi\left(\frac{A}{B}\right)\right]$$

$$\text{Variance} = A^2 + B^2 - \left[B\sqrt{\frac{2}{\pi}}\exp\left(-\frac{A^2}{2\,B^2}\right) + A\,\text{erf}\left(\frac{A}{\sqrt{2}\,B}\right)\right]^2$$

$$\text{Skewness} = \frac{B\sqrt{\frac{2}{\pi}}\exp\left(-\frac{3A^2}{2B^2}\right)\left(4B^2 - \pi\exp\left(\frac{A^2}{B^2}\right)\left(2A^2 + B^2\right)\right)}{\pi\sqrt{\text{Var}^3}} +$$

$$\frac{2A\,\text{erf}\left(\frac{A}{\sqrt{2}B}\right)\left[6B^2\exp\left(-\frac{A^2}{B^2}\right) + 3\sqrt{2\pi}\,A\,B\exp\left(-\frac{A^2}{2B^2}\right)\text{erf}\left(\frac{A}{\sqrt{2}B}\right) + \pi A^2\left(\text{erf}^2\left(\frac{A}{\sqrt{2}B}\right) - 1\right)\right]}{\pi\sqrt{\text{Var}^3}}$$

$$\text{Kurtosis} = -3 +$$

$$\frac{A^4 + 6A^2B^2 + 3B^4 + 6\left(A^2 + B^2\right)\left[B\sqrt{\frac{2}{\pi}}\exp\left(-\frac{A^2}{2B^2}\right) + A\,\text{erf}\left(\frac{A}{\sqrt{2}B}\right)\right]^2}{\text{Var}^2} -$$

$$\frac{3}{\text{Var}^2}\left[B\sqrt{\frac{2}{\pi}}\exp\left(-\frac{A^2}{2B^2}\right) + A\,\text{erf}\left(\frac{A}{\sqrt{2}B}\right)\right]^4 -$$

$$\frac{4\exp\left(-\frac{A^2}{B^2}\right)\left[B\sqrt{\frac{2}{\pi}} + A\exp\left(\frac{A^2}{2B^2}\right)\text{erf}\left(\frac{A}{\sqrt{2}B}\right)\right]\left[B\sqrt{\frac{2}{\pi}}\left(A^2 + 2B^2\right) + A\left(A^2 + 3B^2\right)\exp\left(\frac{A^2}{2B^2}\right)\text{erf}\left(\frac{A}{\sqrt{2}B}\right)\right]}{\text{Var}^2}$$

Mode, Median, Q1, Q3, qMean, qMode: no simple closed form

Notes

1. This distribution is indifferent to the sign of A. Therefore, to avoid ambiguity, A is here restricted to be positive.
2. Mode > 0 when A > B.

Aliases and Special Cases

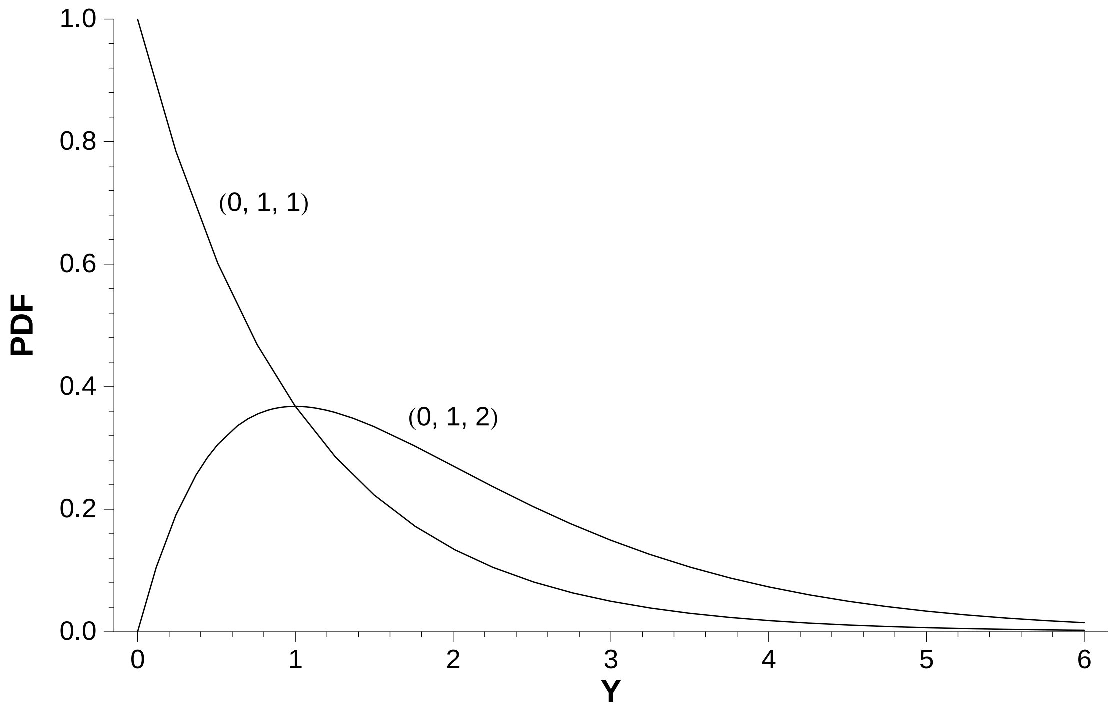
1. If A = 0, the FoldedNormal distribution becomes the **HalfNormal** distribution.
2. The FoldedNormal distribution is identical to the distribution of χ' (*Non-central chi*) with one degree of freedom and non-centrality parameter $(A/B)^2$.

Characterizations

1. If Z ~**Normal**(A, B), |Z| ~FoldedNormal(A, B).

Gamma(A,B,C)

$y > A, \quad B > 0, \quad 0 < C \le 100$



$$\text{PDF} = \frac{1}{B\,\Gamma(C)} \left(\frac{y-A}{B}\right)^{C-1} \exp\left(\frac{A-y}{B}\right)$$

$$\text{CDF} = \Gamma\left(C, \frac{y-A}{B}\right)$$

Parameters -- A (γ): Location, B (β): Scale, C (α): Shape

Moments, etc.

$$\text{Mean} = A + B\,C$$

$$\text{Variance} = B^2\,C$$

$$\text{Skewness} = \frac{2}{\sqrt{C}}$$

$$\text{Kurtosis} = \frac{6}{C}$$

$$\text{Mode} = A + B\,(C-1)$$

Median, Q1, Q3: no simple closed form

$$\text{qMean} = \Gamma\left(C, C\right) \qquad \text{qMode} = \Gamma\left(C, C-1\right)$$

Notes

1. The Gamma distribution is right-skewed.
2. To model a left-skewed distribution, try modeling w = –y.
3. The Gamma distribution approaches a **Normal** distribution in the limit as C goes to infinity.
4. In the literature, C > 0. The restriction shown above is required primarily to recognize when the PDF is not right-skewed.

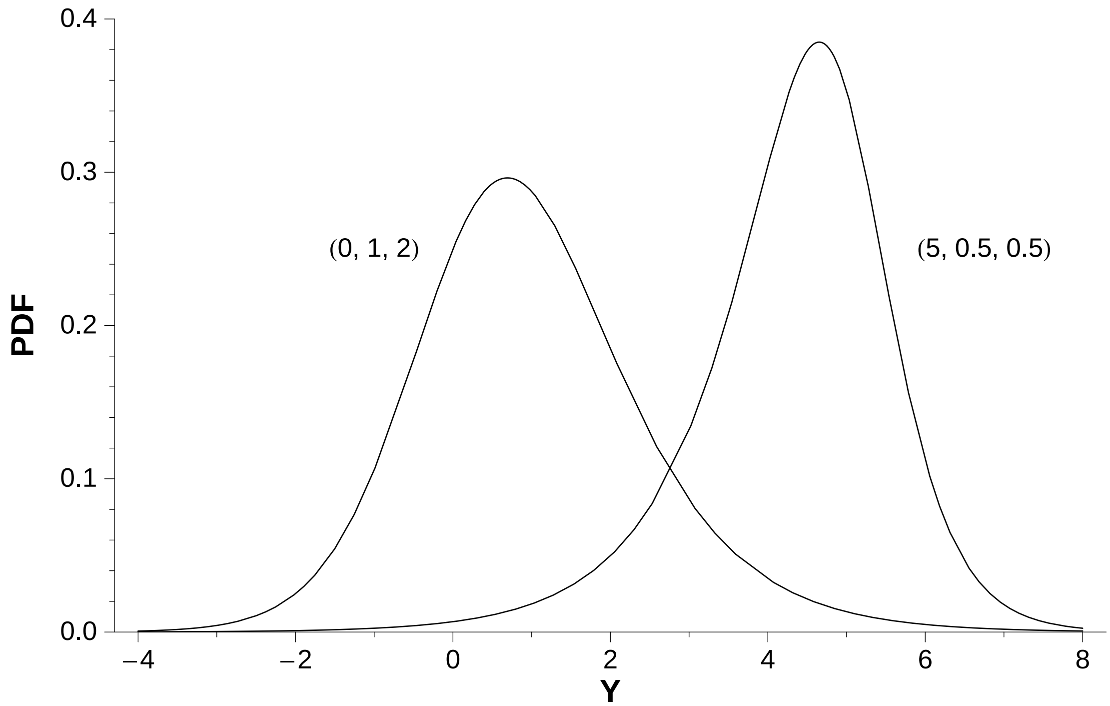
Aliases and Special Cases

1. Gamma(A, B, C), where C is an integer, is the *Erlang* distribution.
2. Gamma(A, B, 1) is the **Exponential** distribution.
3. Gamma(0, 2, ν/2) is the *Chi-square* distribution with ν degrees of freedom.

Characterizations

1. If Z_1 ~Gamma(A, B, C_1) and Z_2 ~Gamma(A, B, C_2), then ($Z_1 + Z_2$) ~Gamma(A, B, $C_1 + C_2$).
2. If $Z_1, Z_2, \ldots, Z_\nu$ ~**Normal**(0, 1), then $W = \sum_{k=1}^{\nu} Z_k^2$ ~Gamma(0, 2, ν/2).
3. If $Z_1, Z_2, \ldots,$ Zn ~**Exponential**(A, B), then $W = \sum_{k=1}^{n} Z_k$ ~Erlang(A, B, n).

GenLogistic(A,B,C) $B, C > 0$



$$\text{PDF} = \frac{C}{B}\frac{\exp\left(\frac{A-y}{B}\right)}{\left[1+\exp\left(\frac{A-y}{B}\right)\right]^{C+1}}$$

$$\text{CDF} = \frac{1}{\left[1+\exp\left(\frac{A-y}{B}\right)\right]^{C}}$$

Parameters -- A: Location, B: Scale, C: Shape

Moments, etc.

$$\text{Mean} = A + \left(\gamma + \psi\left(C\right)\right)B$$

$$\text{Variance} = \left[\frac{\pi^2}{6} + \psi'\left(C\right)\right]B^2$$

$$\text{Skewness} = \frac{\psi''\left(C\right) + 2\,\zeta\left(3\right)}{\left[\frac{\pi^2}{6} + \psi'\left(C\right)\right]^{\frac{3}{2}}}$$

$$\text{Kurtosis} = \frac{12\left[\pi^4 + 15\,\psi'''\left(C\right)\right]}{5\left[\pi^2 + 6\,\psi'\left(C\right)\right]^2}$$

$$\text{Mode} = A + B \log\left(C\right)$$

$$\text{Median} = A - B \log\left(\sqrt[C]{2} - 1\right)$$

$$Q1 = A - B \log\left(\sqrt[C]{4} - 1\right) \qquad Q3 = A - B \log\left(\sqrt[C]{\frac{4}{3}} - 1\right)$$

$$\text{qMean} = \left[1 + \exp\left(-H\left(C - 1\right)\right)\right]^{-C} \qquad \text{qMode} = \left(\frac{C}{C+1}\right)^{C}$$

$$\text{RandVar} = A - B \log\left(\frac{1}{\sqrt[C]{u}} - 1\right)$$

Notes

1. The Generalized Logistic distribution is a generalization of the **Logistic** distribution.
2. The Generalized Logistic distribution is left-skewed when $C < 1$ and right-skewed for $C > 1$.
3. There are additional generalizations of the **Logistic** distribution.

Aliases and Special Cases

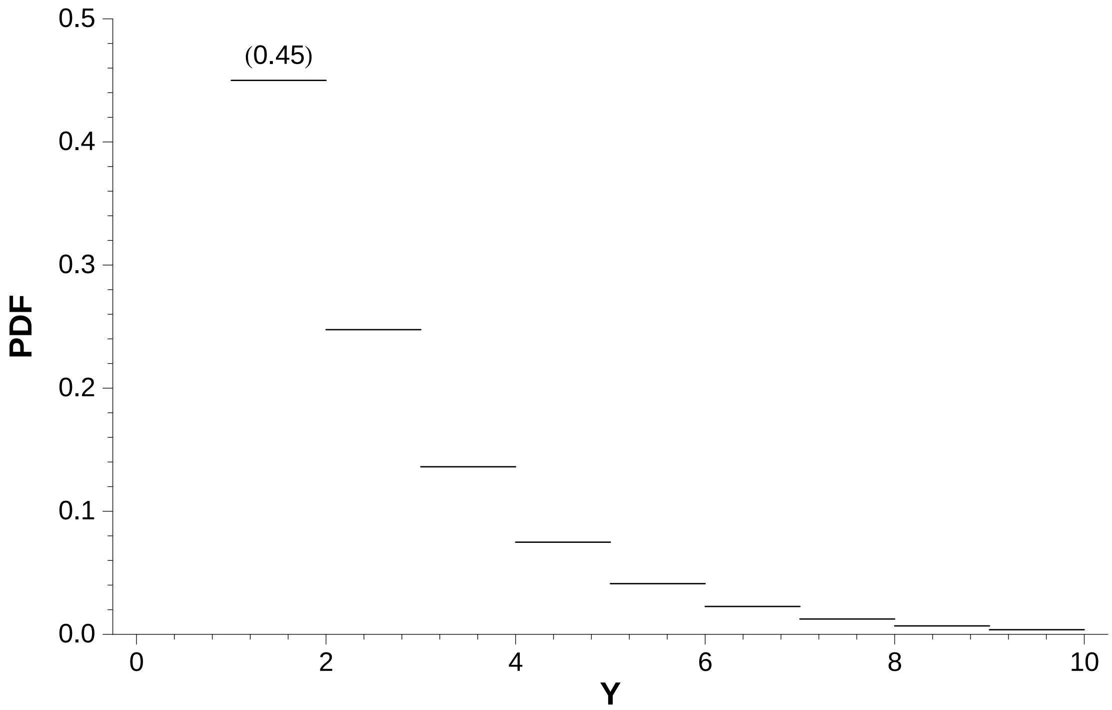
1. The Generalized Logistic distribution becomes the **Logistic** distribution when $C = 1$.

Characterizations

1. The Generalized Logistic has been used in the analysis of extreme values.

Geometric(A)

$y = 1, 2, 3, \ldots, \quad 0 < A < 1$



$$\text{PDF} = A\left(1 - A\right)^{y-1}$$

Parameters -- A (p): Prob(success)

Moments, etc.

$$\text{Mean} = \frac{1}{A}$$

$$\text{Variance} = \frac{1 - A}{A^2}$$

$$\text{Mode} = 1$$

Notes

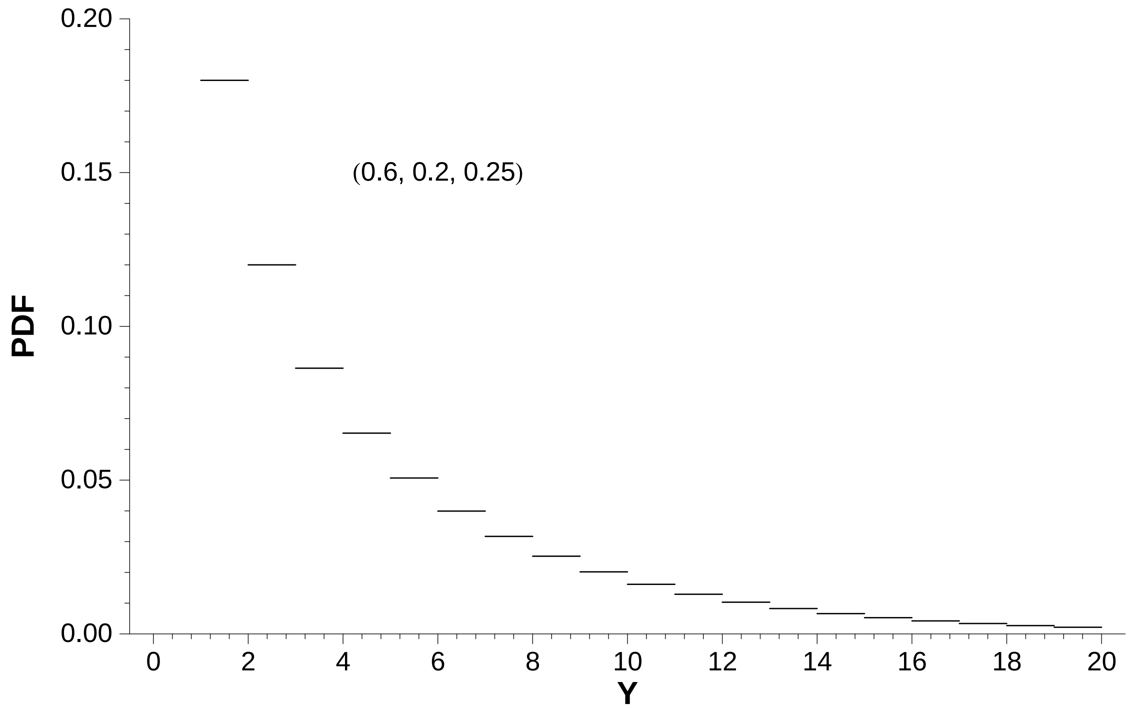
Aliases and Special Cases

1. The Geometric distribution is the discrete version of the **Exponential** distribution.
2. The Geometric distribution is sometimes called the *Furry* distribution.

Characterizations

1. In a series of Bernoulli trials, with Prob(success) = A, the number of trials required to realize the first success is ~Geometric(A).
2. For the B^{th} success, see the **NegativeBinomial** distribution.

Geometric(A)&Geometric(B) $y = 1, 2, 3, \ldots, \quad 0 < B < A < 1, \quad 0 < p < 1$



$$\text{PDF} = p\,A\,(1-A)^{y-1} + (1-p)\,B\,(1-B)^{y-1}$$

Parameters -- A, B (π_1, π_2): Prob(success), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \frac{p}{A} + \frac{1-p}{B}$$

$$\text{Variance} = \frac{A^2\,(1-B) + p\,B\,(A-B)\,(A-2) - p^2\,(A-B)^2}{A^2\,B^2}$$

$$\text{Mode} = 1$$

Notes

1. Here, parameter A is stipulated to be the Component with the larger Prob(success).
2. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
3. **Warning!** Mixtures usually have several local optima.

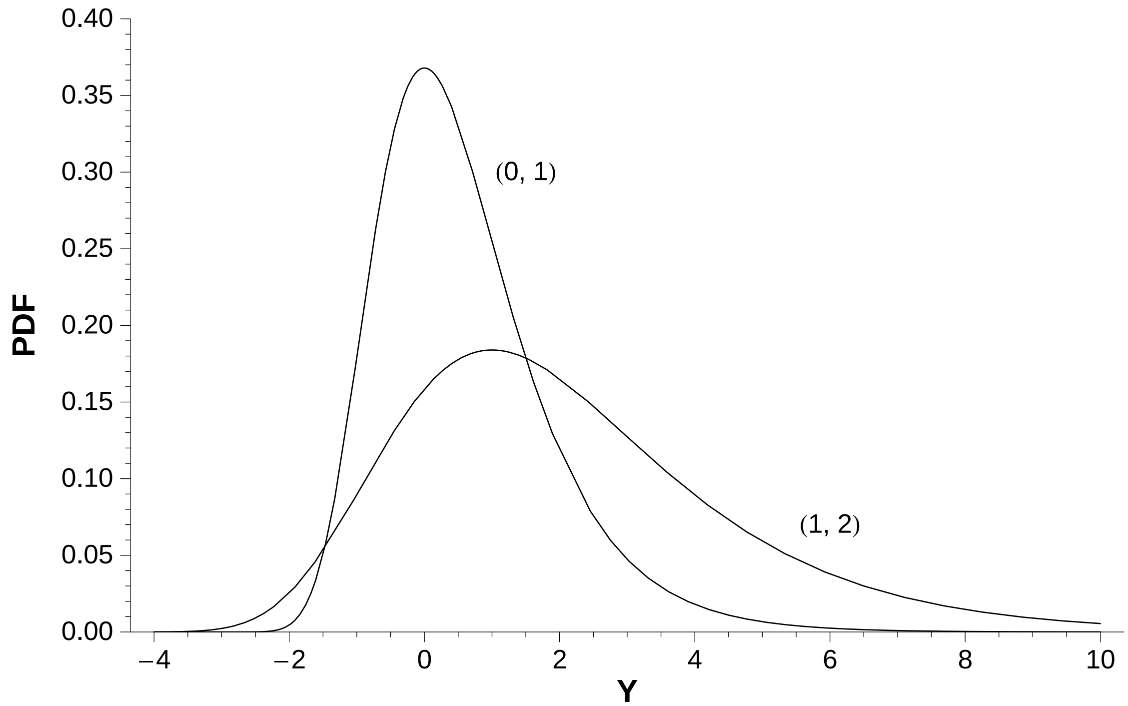
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Gumbel(A,B)

B > 0



$$\text{PDF} = \frac{1}{B}\exp\left(\frac{A-y}{B}\right)\exp\left(-\exp\left(\frac{A-y}{B}\right)\right)$$

$$\text{CDF} = \exp\left(-\exp\left(\frac{A-y}{B}\right)\right)$$

Parameters -- A (ξ): Location, B (θ): Scale

Moments, etc.

$$\text{Mean} = A + \gamma B$$

$$\text{Variance} = \frac{1}{6}(\pi B)^2$$

$$\text{Skewness} = \frac{12\sqrt{6}\,\zeta(3)}{\pi^3} \approx 1.1395$$

$$\text{Kurtosis} = \frac{12}{5}$$

$$\text{Mode} = A$$

$$\text{Median} = A - B \log\left(\log(2)\right)$$

$$Q1 = A - B \log\left(\log(4)\right) \qquad Q3 = A - B \log\left(\log\left(\frac{4}{3}\right)\right)$$

$$\text{qMean} = \exp\left(-\exp(-\gamma)\right) \approx 0.5704 \qquad \text{qMode} = \frac{1}{e} \approx 0.3679$$

$$\text{RandVar} = A - B \log\left(-\log(u)\right)$$

Notes

1. The Gumbel distribution is one of the class of extreme-value distributions.
2. The Gumbel distribution is right-skewed.
3. To model a left-skewed distribution, try modeling w = –y.

Aliases and Special Cases

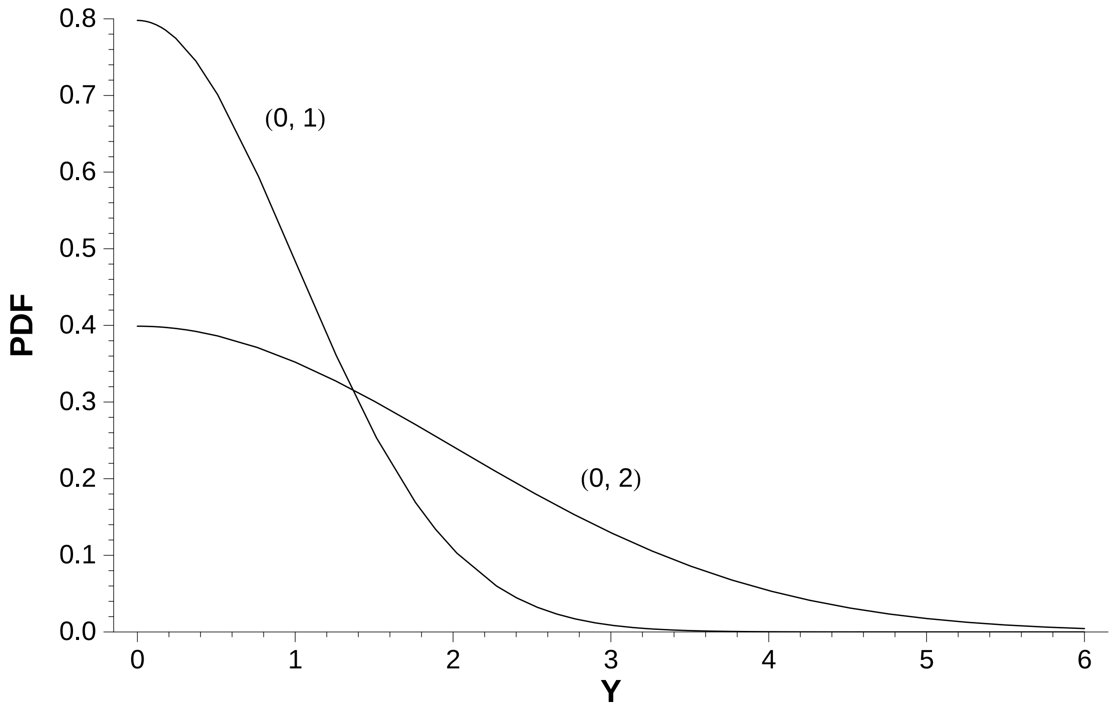
1. The Gumbel distribution is sometimes called the *LogWeibull* distribution.
2. It is also known as the *Gompertz* distribution.
3. It is also known as the *Fisher-Tippett* distribution.

Characterizations

1. Extreme-value distributions are the limiting distributions, as N --> infinity, of the greatest value among N i.i.d. variates selected from a continuous distribution. By replacing y with –y, the smallest values may be modeled.
2. The Gumbel distribution is often used to model maxima when the random variable is unbounded.

HalfNormal(A,B)

$y \geq A, \quad B > 0$



$$\text{PDF} = \frac{1}{B}\sqrt{\frac{2}{\pi}}\,\exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right)$$

$$\text{CDF} = 2\,\Phi\left(\frac{y-A}{B}\right) - 1$$

Parameters -- A (θ): Location, B (λ): Scale

Moments, etc.

$$\text{Mean} = A + B\sqrt{\frac{2}{\pi}}$$

$$\text{Variance} = B^2\left(1 - \frac{2}{\pi}\right)$$

$$\text{Skewness} = \frac{\sqrt{2}\,(4-\pi)}{\sqrt{(\pi-2)^3}} \approx 0.9953$$

$$\text{Kurtosis} = \frac{8\,(\pi-3)}{(\pi-2)^2} \approx 0.8692$$

$$\text{Mode} = A$$

$$\text{Median} \approx A + 0.6745\, B$$

$$\text{Q1} \approx A + 0.3186\, B \qquad \text{Q3} \approx A + 1.150\, B$$

$$\text{qMean} \approx 0.5751 \qquad \text{qMode} = 0$$

Notes

Aliases and Special Cases

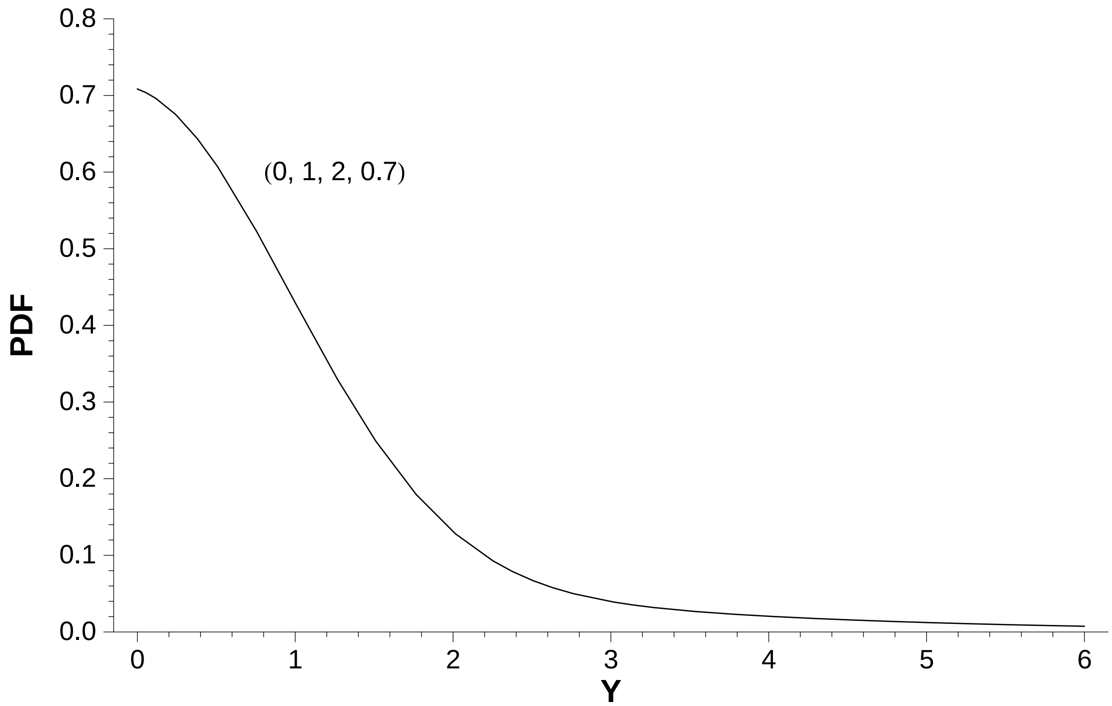
1. The HalfNormal distribution is a special case of both the **Chi** and the **FoldedNormal** distributions.

Characterizations

1. If X ~**Normal**(A, B) is folded (to the right) about its mean, A, the resulting distribution is HalfNormal(A, B).

HNormal(A,B)&Expo(A,C)

$y \geq A, \quad B, C > 0, \quad 0 < p < 1$



$$\text{PDF} = \sqrt{\frac{2}{\pi}}\,\frac{p}{B}\exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{(1-p)}{C}\exp\left(\frac{A-y}{C}\right)$$

$$\text{CDF} = p\ \text{stdHalfNormalCDF}\left(\frac{y-A}{B}\right) + (1-p)\ \text{stdExponentialCDF}\left(\frac{y-A}{C}\right)$$

Parameters -- A (θ): Location, B, C (λ_1, λ_2): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = A + p\,B\sqrt{\frac{2}{\pi}} + (1-p)\,C$$

$$\text{Variance} = \frac{1}{\pi}\left[p\,(\pi - 2\,p)\,B^2 + 2\,p\,(p-1)\,B\,C\,\sqrt{2\,\pi} - (p^2-1)\,\pi\,C^2\right]$$

$$\text{Mode} = A$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.
3. The alternate, **Expo(A,B)&HNormal(A,C)** distribution may be obtained by switching identities in the parameter dialog. In this case, the parameters shown above in the Moments section must be reversed (cf. E&E and H&H).

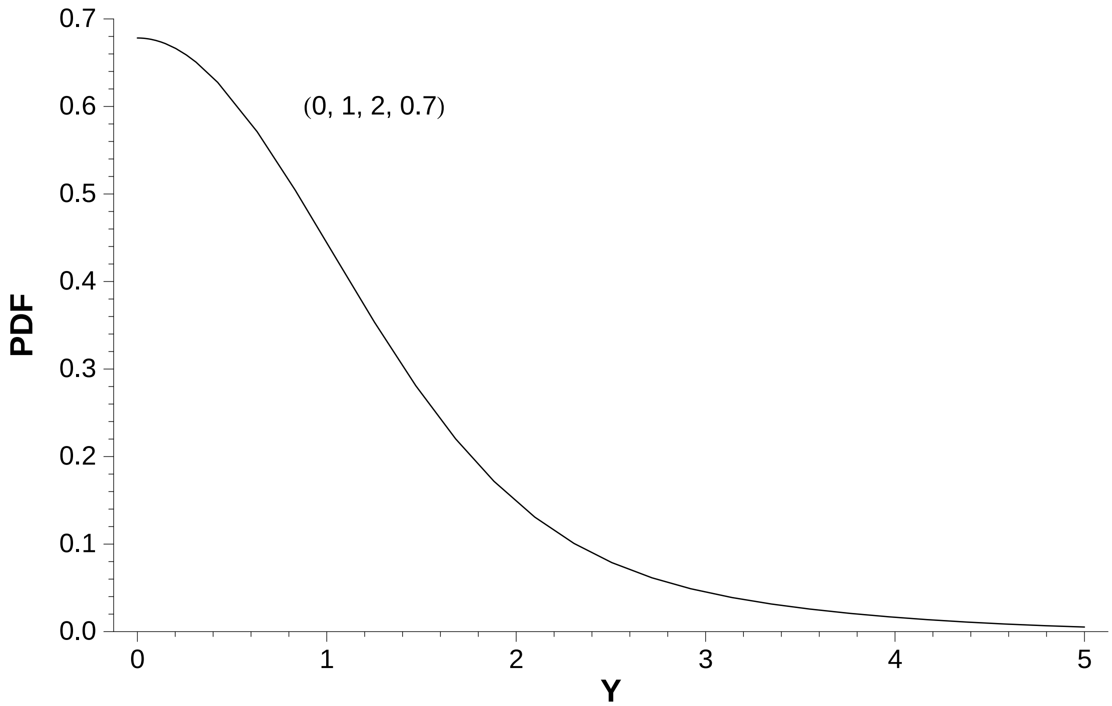
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

HNormal(A,B)&HNormal(A,C)

$y \geq A, \quad B, C > 0, \quad 0 < p < 1$



$$\text{PDF} = \sqrt{\frac{2}{\pi}}\left[\frac{p}{B}\exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{(1-p)}{C}\exp\left(-\frac{1}{2}\left(\frac{y-A}{C}\right)^2\right)\right]$$

$$\text{CDF} = p\ \text{stdHalfNormalCDF}\left(\frac{y-A}{B}\right) + (1-p)\ \text{stdHalfNormalCDF}\left(\frac{y-A}{C}\right)$$

Parameters -- A (θ): Location, B, C (λ_1, λ_2): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = A + \sqrt{\frac{2}{\pi}}\left[p\,B + (1-p)\,C\right]$$

$$\text{Variance} = p\,B^2 + (1-p)\,C^2 - \frac{2}{\pi}\left[p\,B + (1-p)\,C\right]^2$$

$$\text{Mode} = A$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

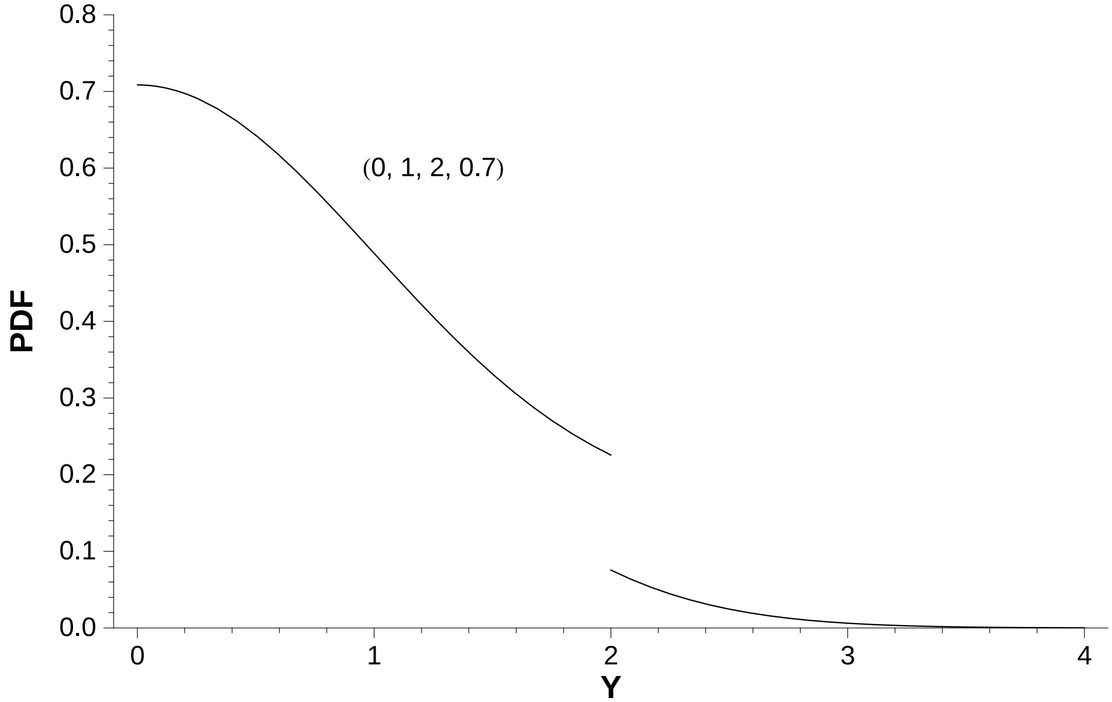
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

HNormal(A,B)&Uniform(A,C)

$A \le y < C, \quad B > 0, \quad 0 < p < 1$



$$\text{PDF} = \sqrt{\frac{2}{\pi}}\,\frac{p}{B}\exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{(1-p)\,(y<C)}{C-A}$$

$$\text{CDF} = \begin{cases} p\ \text{stdHalfNormalCDF}\left(\frac{y-A}{B}\right) + (1-p)\left(\frac{y-A}{C-A}\right), & y < C \\ p\ \text{stdHalfNormalCDF}\left(\frac{y-A}{B}\right) + (1-p), & y \ge C \end{cases}$$

Parameters -- A (θ): Location, B (λ), C : Scale (C = upper bound of Uniform(A,C)), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \frac{1}{2}\left[A + C + p\left(A - C + 2\,B\sqrt{\frac{2}{\pi}}\right)\right]$$

$$\text{Variance} = \frac{12\,p\,B^2\,(\pi - 2\,p) + 12\,p\,B\,(A-C)\,(1-p)\,\sqrt{2\,\pi} + \pi\,(A-C)^2\,(1-p)\,(1+3\,p)}{12\,\pi}$$

$$\text{Mode} = A$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

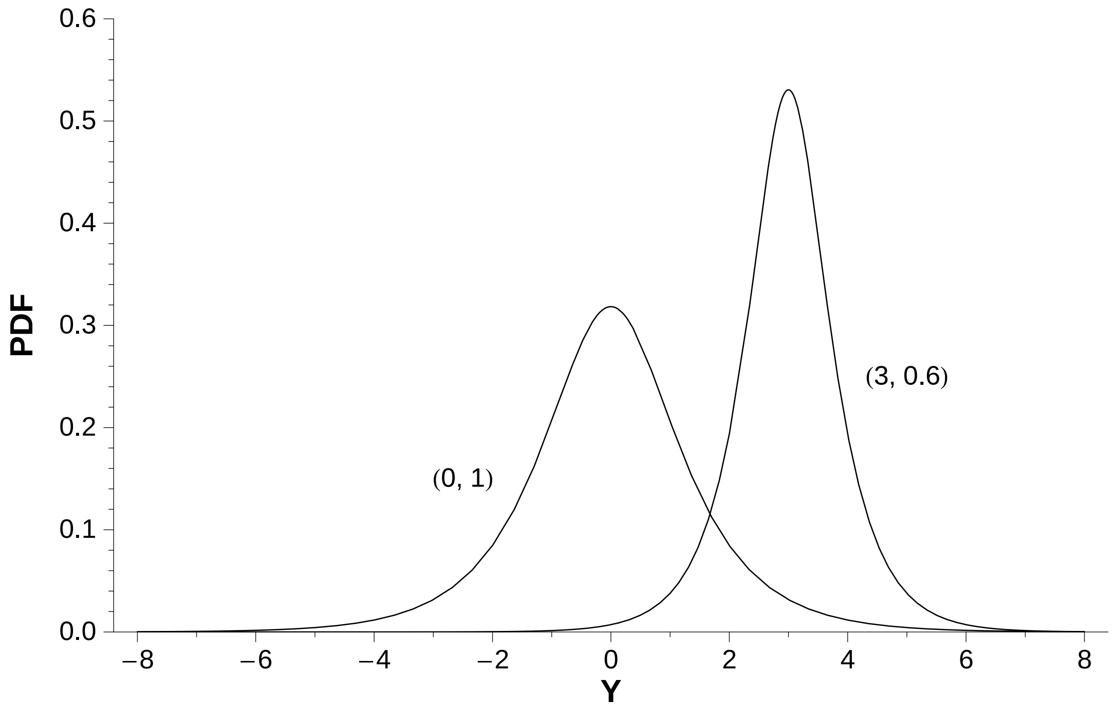
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

HyperbolicSecant(A,B)

B > 0



$$\text{PDF} = \frac{\operatorname{sech}\left(\frac{y - A}{B}\right)}{\pi B}$$

$$\text{CDF} = \frac{2}{\pi} \tan^{-1}\left[\exp\left(\frac{y - A}{B}\right)\right]$$

Parameters -- A: Location, B: Scale

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = \frac{1}{4}(\pi B)^2$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = 2$$

$$Q1 = A - B \log\left(1 + \sqrt{2}\right) \qquad Q3 = A + B \log\left(1 + \sqrt{2}\right)$$

$$\text{qMean} = \text{qMode} = 0.5$$

$$\text{RandVar} = A + B \log\left[\tan\left(\frac{\pi\, u}{2}\right)\right]$$

Notes

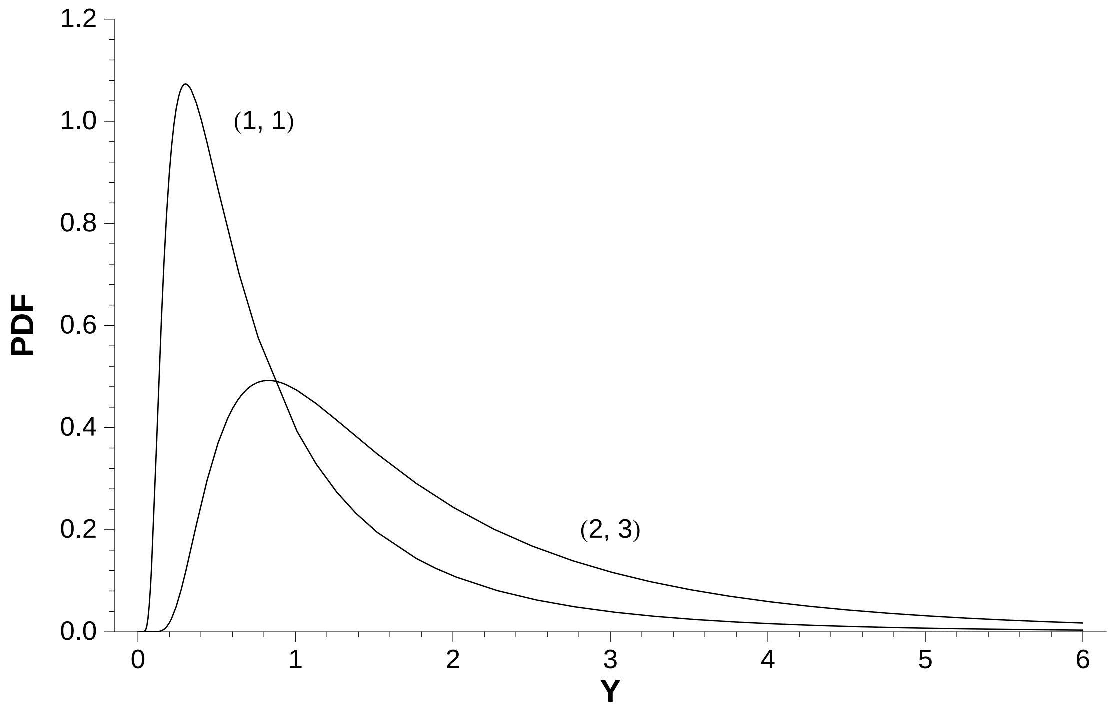
1. The Hyperbolic Secant is related to the **Logistic** distribution.

Aliases and Special Cases

Characterizations

1. If Z ~Hyperbolic Secant, then W = exp(Z) ~Half Cauchy.
2. The Hyperbolic Secant distribution is used in lifetime analysis.

InverseNormal(A,B) $y > 0, \quad A, B > 0$



$$\text{PDF} = \sqrt{\frac{B}{2\,\pi\,y^3}}\,\exp\left(-\frac{B}{2\,y}\left(\frac{y-A}{A}\right)^2\right)$$

$$\text{CDF} = \Phi\left(\sqrt{\frac{B}{y}}\,\frac{y-A}{A}\right) + \exp\left(\frac{2\,B}{A}\right)\Phi\left(\sqrt{\frac{B}{y}}\,\frac{-y-A}{A}\right)$$

Parameters -- A (μ): Location, B (λ): Scale

Moments, etc.

$$\text{Mean} = A$$

$$\text{Variance} = \frac{A^3}{B}$$

$$\text{Skewness} = 3\sqrt{\frac{A}{B}}$$

$$\text{Kurtosis} = \frac{15\,A}{B}$$

$$\text{Mode} = \frac{A}{2\,B}\left(\sqrt{9\,A^2 + 4\,B^2} - 3\,A\right)$$

Median, Q1, Q3, qMean, qMode: no simple closed form

Notes

1. There are several alternate forms for the PDF, some of which have more than two parameters.

Aliases and Special Cases

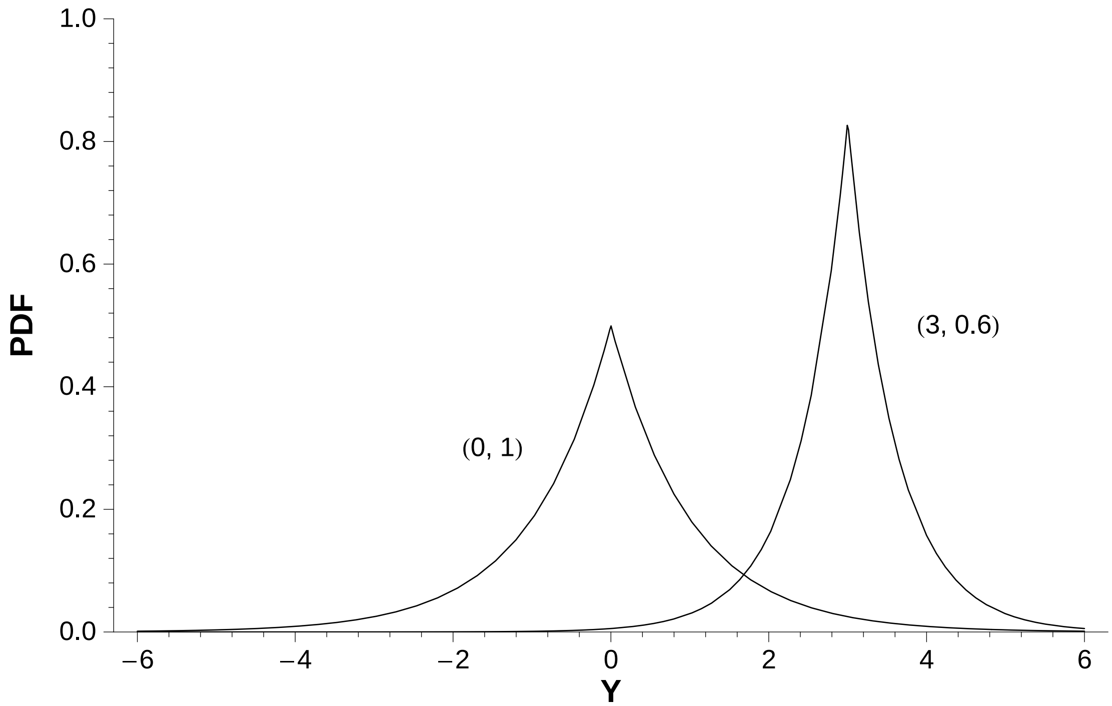
1. The InverseNormal distribution is often called the *Inverse Gaussian* distribution.
2. It is also known as the *Wald* distribution.

Characterizations

1. If a particle, moving in one-dimension with constant speed, exhibits linear Brownian motion, the time required to cover a given distance, d, is ~InverseNormal.

Laplace(A,B)

B > 0



$$\text{PDF} = \frac{1}{2\,B}\exp\left(-\frac{|y-A|}{B}\right)$$

$$\text{CDF} = \begin{cases} \frac{1}{2}\exp\left(\frac{y-A}{B}\right), & y \le A \\ 1-\frac{1}{2}\exp\left(\frac{A-y}{B}\right), & y > A \end{cases}$$

Parameters -- A (θ): Location, B (λ): Scale

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = 2\,B^2$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = 3$$

$$Q1 = A - B\log(2) \qquad Q3 = A + B\log(2)$$

$$\text{qMean} = \text{qMode} = 0.5$$

$$\text{RandVar} = A - B \log(u), \text{ with a random sign}$$

Notes

Aliases and Special Cases

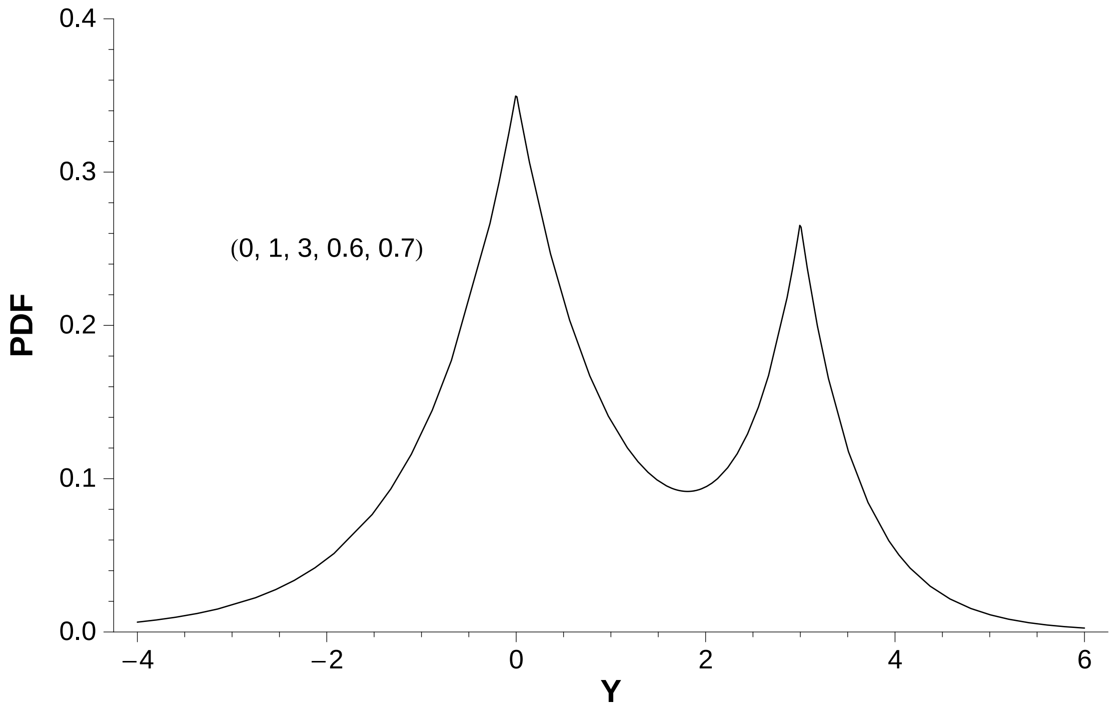
1. The Laplace distribution is often called the *double-exponential* distribution.
2. It is also known as the *bilateral exponential* distribution.

Characterizations

1. The Laplace distribution is the signed analogue of the **Exponential** distribution.
2. Errors of real-valued observations are often ~Laplace or ~**Normal**.

Laplace(A,B)&Laplace(C,D)

$B, D > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{2B} \exp\left(-\frac{|y - A|}{B}\right) + \frac{(1 - p)}{2D} \exp\left(-\frac{|y - C|}{D}\right)$$

$$\text{CDF} = p \ \text{stdLaplaceCDF}\left(\frac{y - A}{B}\right) + (1 - p) \ \text{stdLaplaceCDF}\left(\frac{y - C}{D}\right)$$

Parameters -- A, C (μ_1, μ_2): Location, B, D (λ_1, λ_2): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = p\,A + (1 - p)\,C$$

$$\text{Variance} = p\left[2\,B^2 - (p - 1)(A - C)^2\right] - 2\,(p - 1)\,D^2$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

Aliases and Special Cases

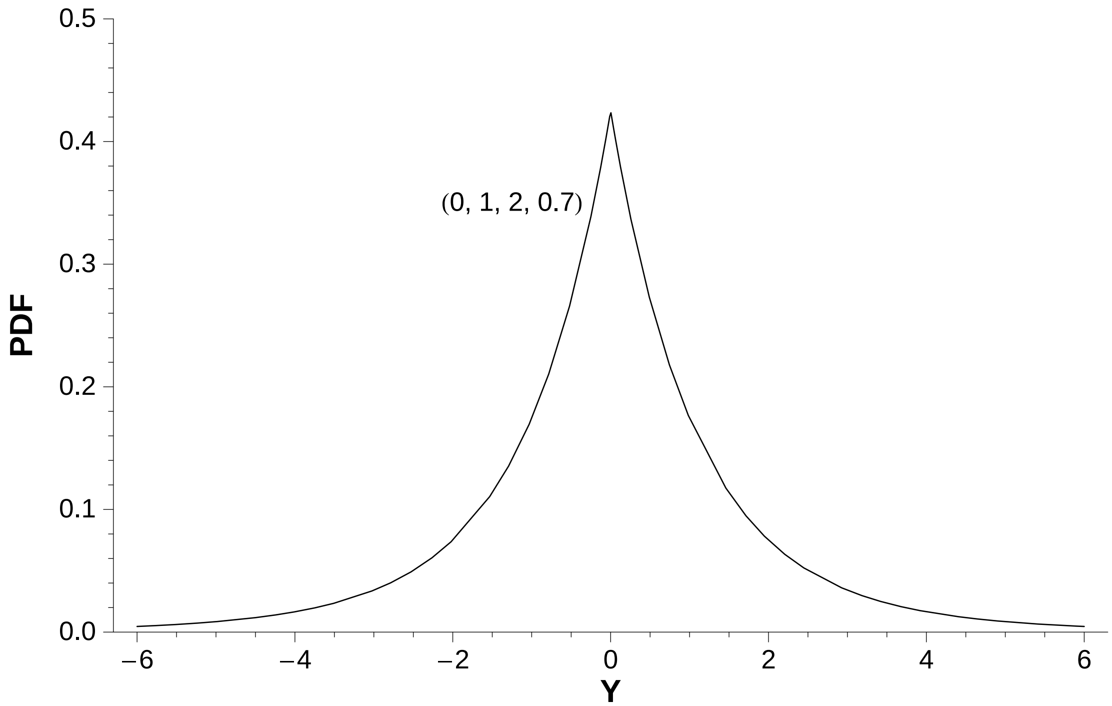
1. This binary mixture is very often referred to as the *Double double-exponential* distribution.

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Laplace(A,B)&Laplace(A,C)

$B, C > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{2\,B} \exp\left(-\frac{|y-A|}{B}\right) + \frac{(1-p)}{2\,C} \exp\left(-\frac{|y-A|}{C}\right)$$

$$\text{CDF} = p \, \text{stdLaplaceCDF}\left(\frac{y-A}{B}\right) + (1-p) \, \text{stdLaplaceCDF}\left(\frac{y-A}{C}\right)$$

Parameters -- A (μ): Location, B, C (λ_1, λ_2): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = 2\left[p\,B^2 + (1-p)\,C^2\right]$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = \frac{6\left(p\,B^4 + (1-p)\,C^4\right)}{\left(p\,B^2 + (1-p)\,C^2\right)^2} - 3$$

Q1, Q3: no simple closed form

qMean = qMode = 0.5

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

Aliases and Special Cases

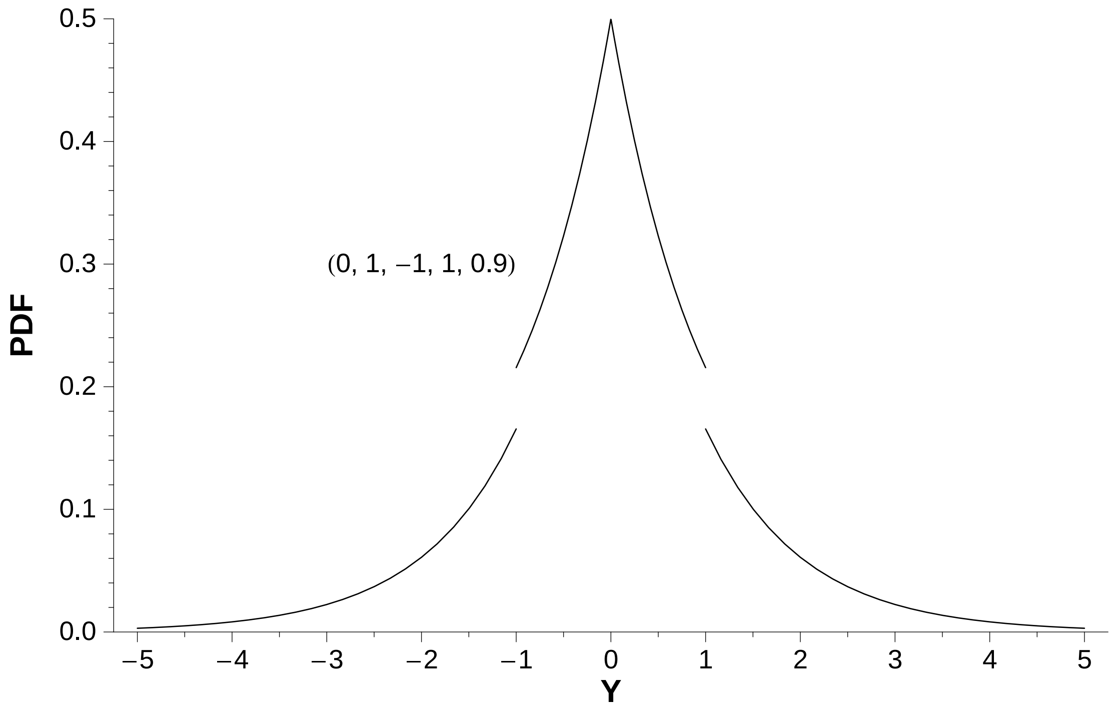
1. This is a special case of the **Laplace(A,B)&Laplace(C,D)** distribution.

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Laplace(A,B)&Uniform(C,D)

$B > 0, \quad C < D, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{2\,B} \exp\left(-\frac{\left|y - A\right|}{B}\right) + \frac{(1-p)\,(C < y < D)}{D - C}$$

$$\text{CDF} = \begin{cases} p \text{ stdLaplaceCDF}\left(\frac{y-A}{B}\right), & y \le C \\ p \text{ stdLaplaceCDF}\left(\frac{y-A}{B}\right) + (1-p)\left(\frac{y-C}{D-C}\right), & C < y < D \\ p \text{ stdLaplaceCDF}\left(\frac{y-A}{B}\right) + (1-p), & y \ge D \end{cases}$$

Parameters -- A (μ), C: Location, B (λ), D : Scale (D = upper bound of Uniform(C,D)), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = p\,A + \frac{(1-p)\,(C+D)}{2}$$

$$\text{Variance} = p\left(A^2 + 2\,B^2\right) - \left[p\,A + \frac{(1-p)\,(C+D)}{2}\right]^2 + \frac{1}{3}(1-p)\left(C^2 + C\,D + D^2\right)$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

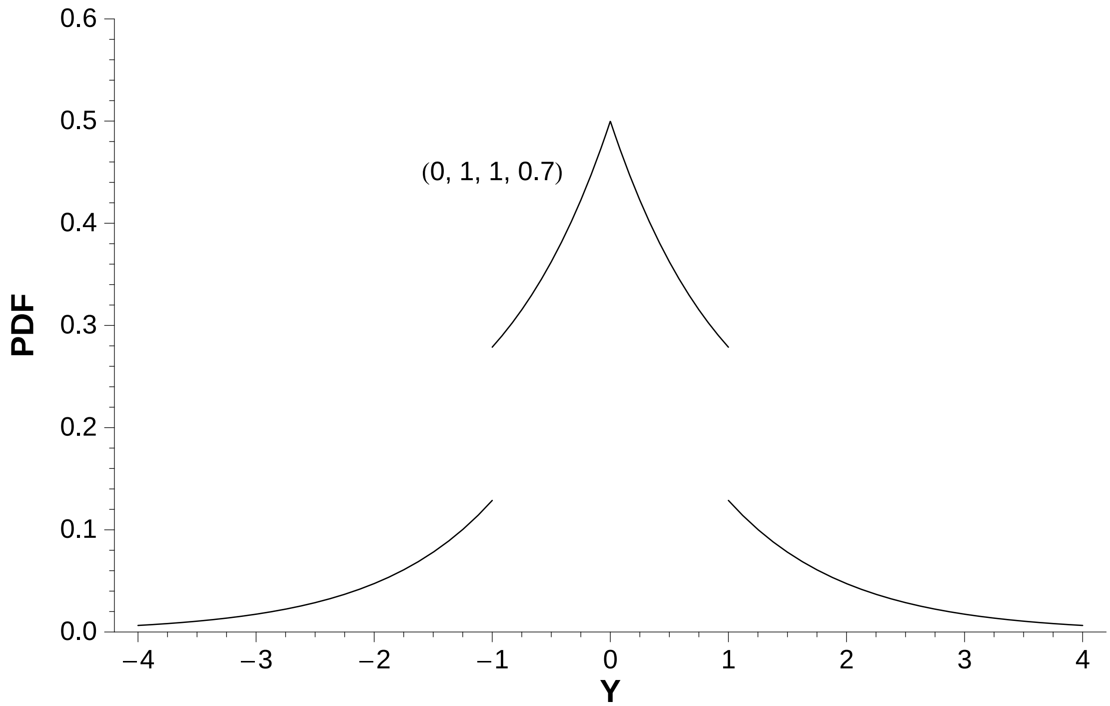
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Laplace(A,B)&Uniform(A,C)

$B, C > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{2\,B} \exp\left(-\frac{\left|y - A\right|}{B}\right) + \frac{\left(1 - p\right)\left[\left(A - C\right) < y < \left(A + C\right)\right]}{2\,C}$$

$$\text{CDF} = \begin{cases} p\ \text{stdLaplaceCDF}\left(\frac{y - A}{B}\right), & y \le A - C \\ p\ \text{stdLaplaceCDF}\left(\frac{y - A}{B}\right) + \left(1 - p\right)\left(\frac{y - A + C}{2\,C}\right), & A - C < y < A + C \\ p\ \text{stdLaplaceCDF}\left(\frac{y - A}{B}\right) + \left(1 - p\right), & y \ge A + C \end{cases}$$

Parameters -- A (μ): Location, B (λ), C : Scale (C = half-width of Uniform(A,C)), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = \frac{1}{3}\left[6\,p\,B^2 + \left(1 - p\right)C^2\right]$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = \frac{9\left(120\,p\,B^4 + (1-p)\,C^4\right)}{5\left(6\,p\,B^2 + (1-p)\,C^2\right)^2} - 3$$

Q1, Q3: no simple closed form

qMean = qMode = 0.5

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.
3. Note that the parameters for the Uniform component are different from those in the **Uniform**(A,B) distribution. Here, A is the Mean and C is the half-width.

Aliases and Special Cases

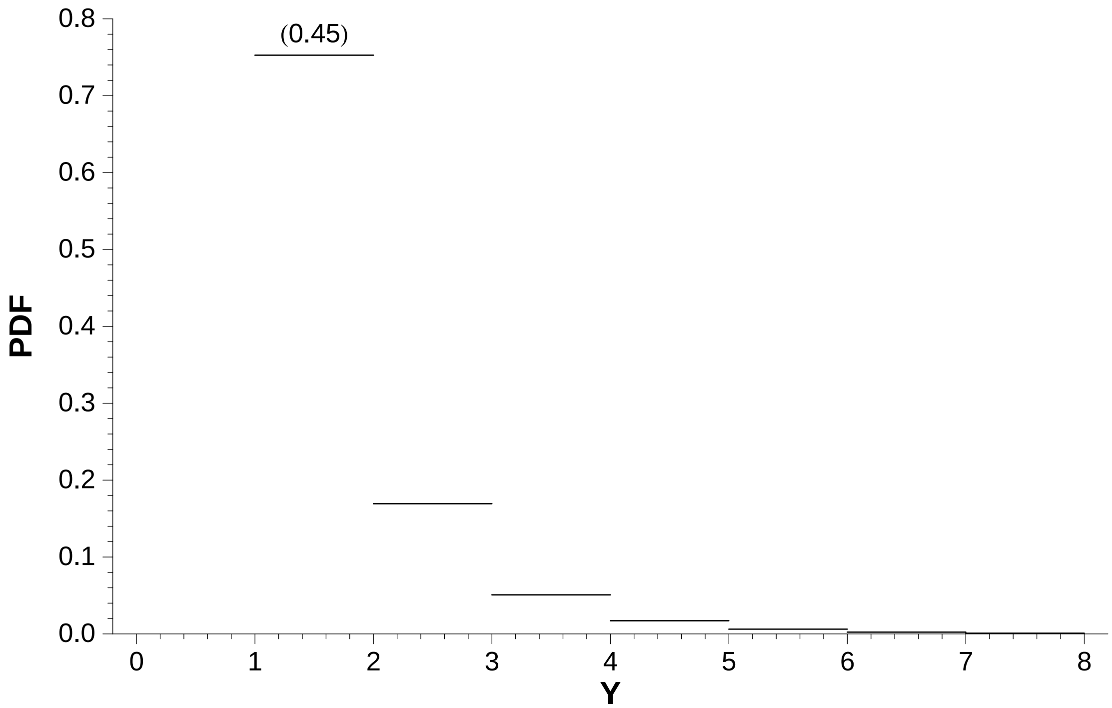
1. This is a special case of the **Laplace(A,B)&Uniform(C,D)** distribution.

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Logarithmic(A)

$y = 1, 2, 3, \ldots, \quad 0 < A < 1$



$$\text{PDF} = \frac{-A^y}{\log(1-A)\,y}$$

Parameters -- A (θ): Shape

Moments, etc.

$$\text{Mean} = \frac{-A}{(1-A)\log(1-A)}$$

$$\text{Variance} = \frac{-A\left[A + \log(1-A)\right]}{(1-A)^2 \log^2(1-A)}$$

$$\text{Mode} = 1$$

Notes

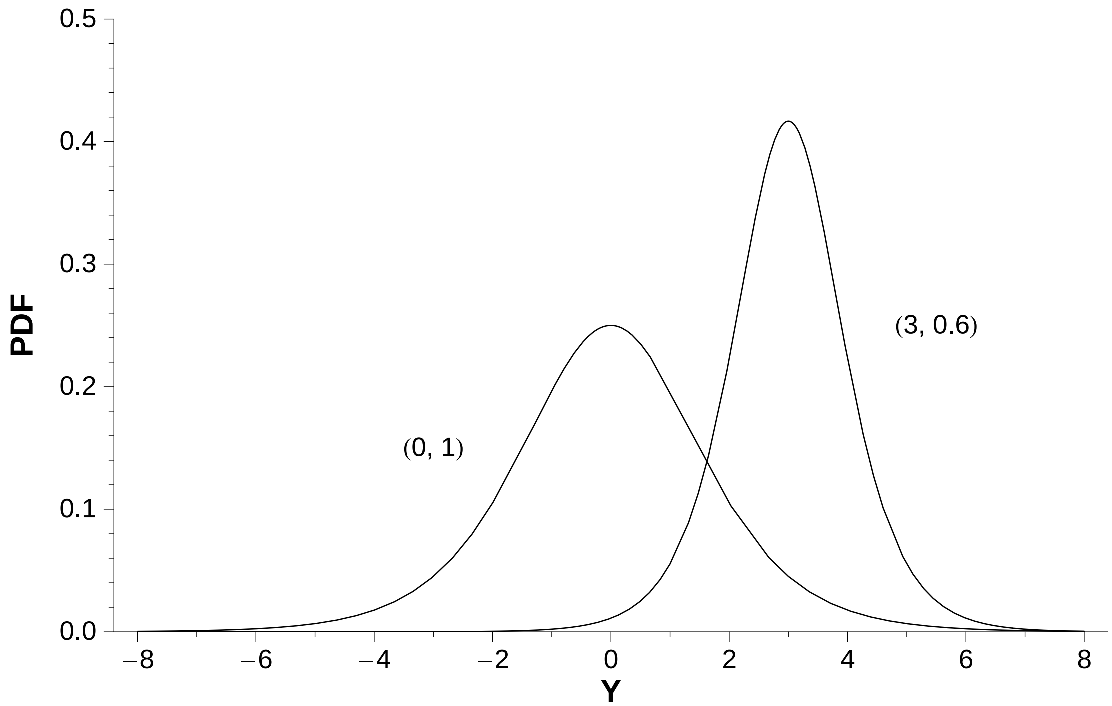
Aliases and Special Cases

Characterizations

1. The Logarithmic distribution has been used to model the numbers of items of a product purchased by a buyer in a given period of time.

Logistic(A,B)

B > 0



$$\text{PDF} = \frac{1}{B} \frac{\exp\left(\frac{y-A}{B}\right)}{\left[1+\exp\left(\frac{y-A}{B}\right)\right]^2}$$

$$\text{CDF} = \frac{1}{1+\exp\left(\frac{A-y}{B}\right)}$$

Parameters -- A (α): Location, B (β): Scale

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = \frac{1}{3}(\pi B)^2$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = \frac{6}{5}$$

$$Q1 = A - B \log(3) \qquad Q3 = A + B \log(3)$$

$$qMean = qMode = 0.5$$

$$RandVar = A + B \log\left(\frac{u}{1 - u}\right)$$

Notes

1. The logistic distribution is often used as an approximation to other symmetrical distributions due to the mathematical tractability of its CDF.

Aliases and Special Cases

1. The logistic distribution is sometimes called the *Sech-squared* distribution.

Characterizations

1. The *logistic law of growth* is described by the following differential equation:

$$\frac{dP}{dt} = \alpha P - \beta P^2$$

whence

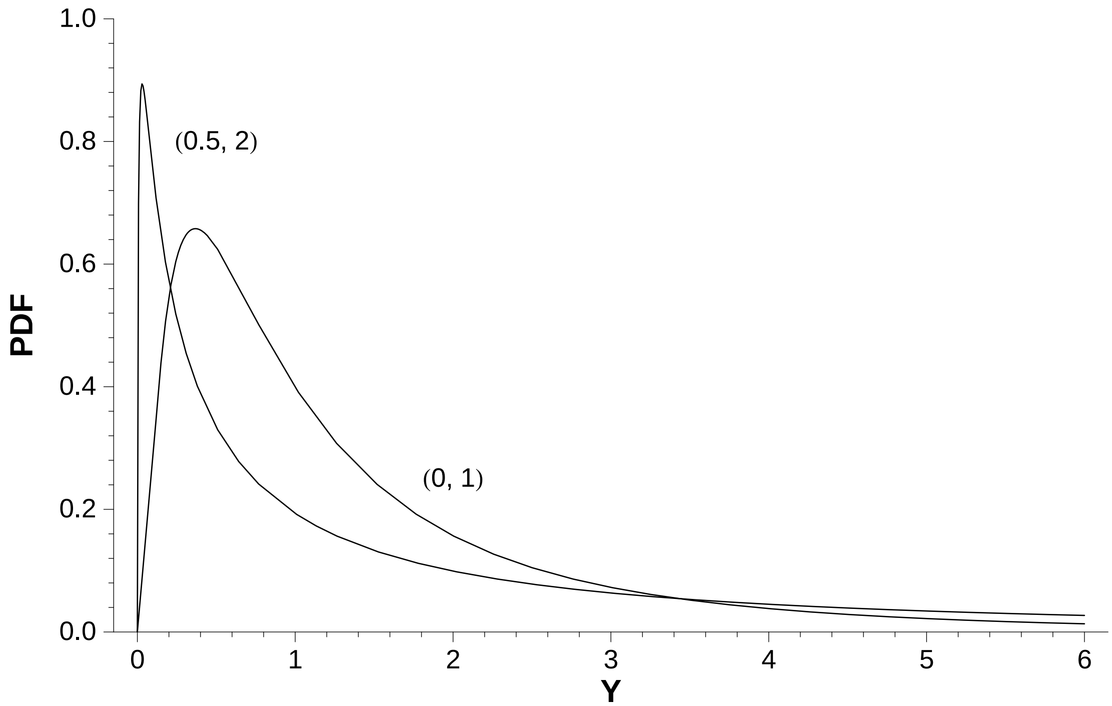
$$P(t) = \frac{P_0 P_\infty}{P_0 + (P_\infty - P_0) \exp(-\alpha t)}$$

where P_0 is the initial and $P_\infty = \frac{\alpha}{\beta}$ the final population size. This sigmoidal function, P(t), reduces to exponential growth when $\alpha >> \beta$. After appropriate normalization, it becomes the CDF given above.

2. If lo and hi are the minima and maxima of a random sample (size = N) then, as N --> infinity, the asymptotic distribution of the midrange = (hi – lo)/2 is ~Logistic.

LogNormal(A,B)

$y > 0, \quad B > 0$



$$\text{PDF} = \frac{1}{B\, y \sqrt{2\,\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(y) - A}{B}\right)^2\right)$$

$$\text{CDF} = \Phi\left(\frac{\log(y) - A}{B}\right)$$

Parameters -- A (ζ): Location, B (σ): Scale, both measured in log space

Moments, etc.

$$\text{Mean} = \exp\left(A + \frac{B^2}{2}\right)$$

$$\text{Variance} = \exp\left(2\,A + B^2\right)\left(\exp\left(B^2\right) - 1\right)$$

$$\text{Skewness} = \left(e + 2\right)\sqrt{e - 1} \approx 6.1849, \text{ for } A = 0 \text{ and } B = 1$$

$$\text{Kurtosis} = e^4 + 2\,e^3 + 3\,e^2 - 6 \approx 110.94, \text{ for } A = 0 \text{ and } B = 1$$

$$\text{Mode} = \exp\left(A - B^2\right)$$

$$\text{Median} = \exp(A)$$

$$Q1 \approx \exp(A - 0.6745\, B) \qquad Q3 \approx \exp(A + 0.6745\, B)$$

qMean, qMode: no simple closed form

Notes

1. The LogNormal distribution is always right-skewed.
2. There are several alternate forms for the PDF, some of which have more than two parameters.
3. Parameters A and B are the mean and standard deviation of y in (natural) log space. Therefore, their units are similarly transformed.

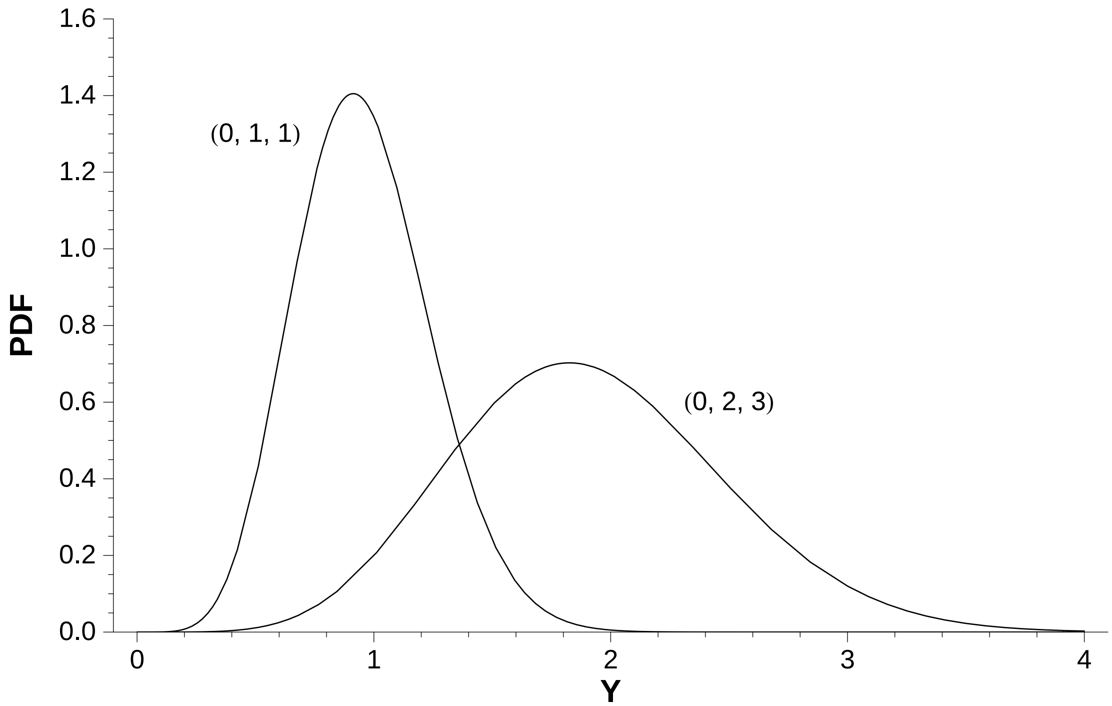
Aliases and Special Cases

1. The LogNormal distribution is sometimes called the *Cobb-Douglas* distribution, especially when applied to econometric data.
2. It is also known as the *antilognormal* distribution.

Characterizations

1. As the PDF suggests, the LogNormal distribution is the distribution of a random variable which, in log space, is ~**Normal**.

Nakagami(A,B,C) $y > A, \quad B > 0, \quad 0 < C \le 100$



$$\text{PDF} = \frac{2\,C^{C}}{B\,\Gamma(C)}\left(\frac{y-A}{B}\right)^{2C-1}\exp\left(-C\left(\frac{y-A}{B}\right)^{2}\right)$$

$$\text{CDF} = \Gamma\left(C, C\left(\frac{y-A}{B}\right)^{2}\right)$$

Parameters -- A: Location, B: Scale, C (ν): Shape (also, degrees of freedom)

Moments, etc.

$$\text{Mean} = A + B\,\frac{\Gamma\left(C+\frac{1}{2}\right)}{\sqrt{C}\,\Gamma(C)}$$

$$\text{Variance} = \left[1 - \frac{\Gamma^{2}\left(C+\frac{1}{2}\right)}{C\,\Gamma^{2}(C)}\right]B^{2}$$

$$\text{Skewness} = \frac{2\,\Gamma^3\left(C+\frac{1}{2}\right) + \Gamma^2(C)\left[\Gamma\left(C+\frac{3}{2}\right) - 3\,C\,\Gamma\left(C+\frac{1}{2}\right)\right]}{\Gamma^3(C)\left[C\left(1 - \frac{C\,\Gamma^2\left(C+\frac{1}{2}\right)}{\Gamma^2(C+1)}\right)\right]^{\frac{3}{2}}}$$

$$\text{Kurtosis} = \frac{-6\,\Gamma^4\left(C+\frac{1}{2}\right) - 3\,C^2\,\Gamma^4(C) + \Gamma^3(C)\,\Gamma(C+2) + 2^{3-4C}(4\,C-1)\,\pi\,\Gamma^2(2\,C)}{\left[\Gamma^2\left(C+\frac{1}{2}\right) - C\,\Gamma^2(C)\right]^2}$$

$$\text{Mode} = A + B\sqrt{\frac{2\,C-1}{2\,C}}$$

Quantiles, etc.: no simple closed form

Notes

1. In the literature, $C > 0$. The restrictions shown above are required for convergence when the data are left-skewed and to ensure the existence of a Mode.

Aliases and Special Cases

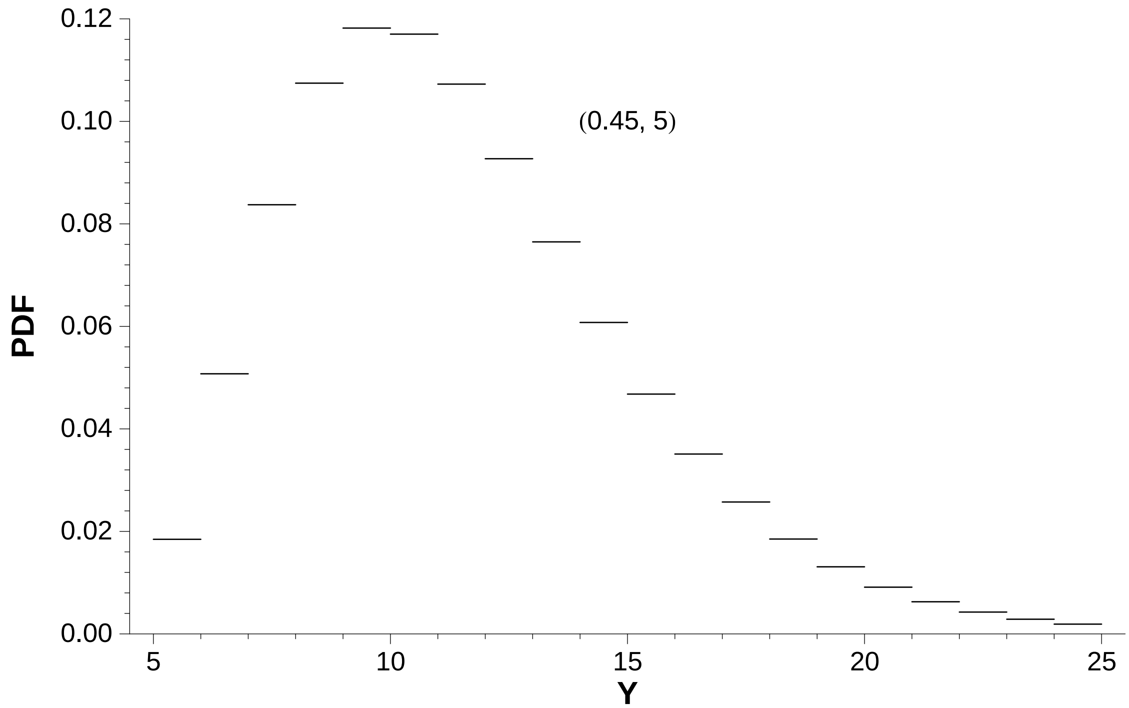
1. cf. **Chi** distribution.

Characterizations

1. The Nakagami distribution is a generalization of the **Chi** distribution.

NegativeBinomial(A,B)

$y = 1, 2, 3, \ldots, \quad 0 < A < 1, \quad 1 \le B \le y$



$$\text{PDF} = \binom{y-1}{B-1} A^{B} \left(1-A\right)^{y-B}$$

Parameters -- A (p): Prob(success), B (k): a constant, target number of successes

Moments, etc.

$$\text{Mean} = \frac{B}{A}$$

$$\text{Variance} = \frac{B\left(1-A\right)}{A^{2}}$$

$$\text{Mode} = \text{int}\left(\frac{A+B-1}{A}\right)$$

Notes

1. Although not supported here, the NegativeBinomial distribution may be generalized to include non-integer values of B.
2. If (B – 1)/A is an integer, Mode also equals (B – 1)/A .
3. *Regress+* requires B to be Constant.

Aliases and Special Cases

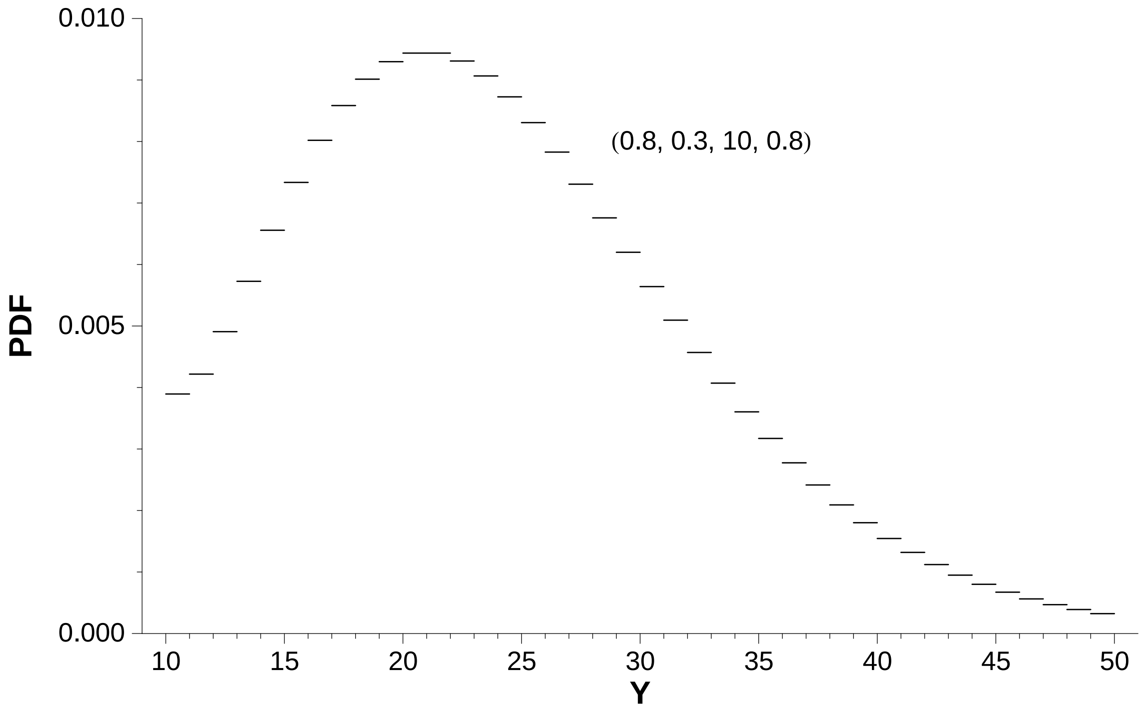
1. The NegativeBinomial is also known as the *Pascal* distribution. The latter name is restricted to the case (as here) in which B is an integer.
2. It is also known as the Polya distribution.
3. If B = 1, the NegativeBinomial distribution becomes the **Geometric** distribution.

Characterizations

1. If Prob(success) = A, the number of Bernoulli trials required to realize the B^{th} success is ~NegativeBinomial(A, B).

NegBinomial(A,C)&NegBinomial(B,C)

$y = 1, 2, 3, \ldots, \quad 0 < B < A < 1, \quad 1 \le C \le y, \quad 0 < p < 1$



$$\text{PDF} = \binom{y-1}{C-1}\left[p\,A^{C}\left(1-A\right)^{y-C} + \left(1-p\right)B^{C}\left(1-B\right)^{y-C}\right]$$

Parameters -- A, B (π_1, π_2): Prob(success), C (k): a constant, target number of successes, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = C\left[\frac{p}{A} + \frac{\left(1-p\right)}{B}\right]$$

$$\text{Variance} = \frac{C\left[p\,B^2\left(1 + C\left(1-p\right)\right) - A^2\left(1-p\right)\left(B - p\,C - 1\right) - p\,A\,B\left(B + 2\,C\left(1-p\right)\right)\right]}{A^2\,B^2}$$

Mode: no simple closed form

Notes

1. Here, parameter A is stipulated to be the Component with the larger Prob(success).
2. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
3. **Warning!** Mixtures usually have several local optima.

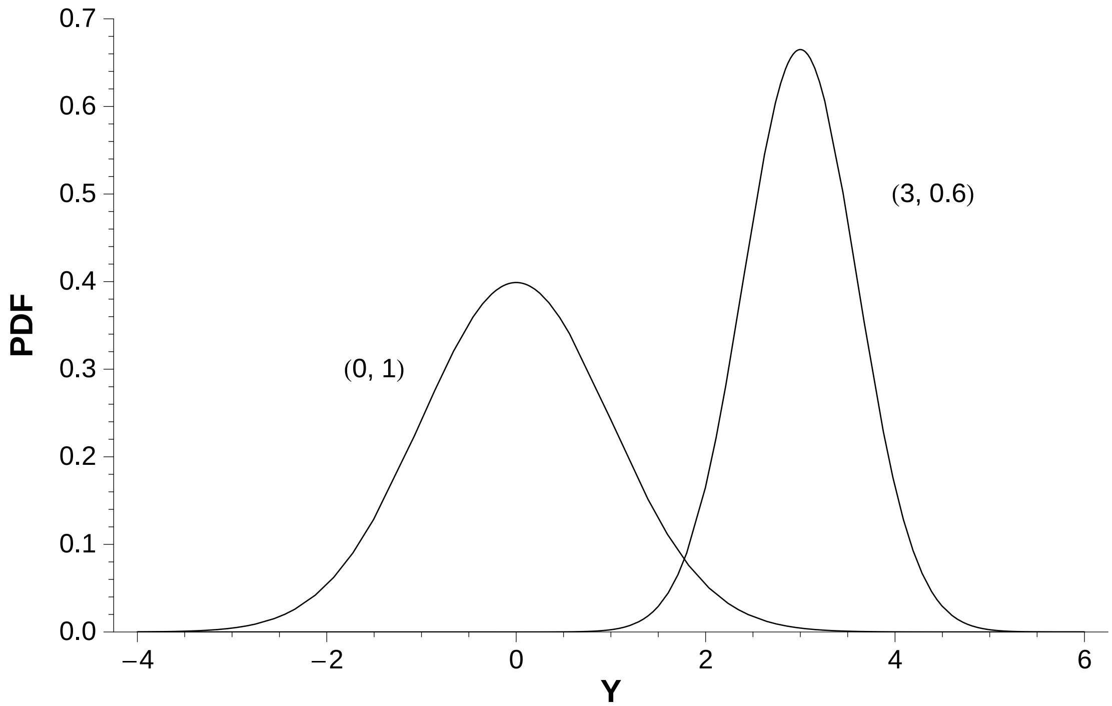
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Normal(A,B)

$B > 0$



$$\text{PDF} = \frac{1}{B\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right)$$

$$\text{CDF} = \Phi\left(\frac{y-A}{B}\right)$$

Parameters -- A (μ): Location, B (σ): Scale

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = B^2$$

$$\text{Skewness} = \text{Kurtosis} = 0$$

$$Q1 \approx A - 0.6745\,B \qquad Q3 \approx A + 0.6745\,B$$

$$\text{qMean} = \text{qMode} = 0.5$$

Notes

1. The CDF is generally tabulated in terms of the standard variable z:

$$z = \frac{y - A}{B}$$

2. The sample standard deviation, s, is the maximum-likelihood estimator of B but is biased with respect to the population value. The latter may be estimated as follows:

$$B = \sqrt{\frac{N}{N-1}}\, s$$

   where N is the sample size.

Aliases and Special Cases

1. The Normal distribution is very often called the *Gaussian* distribution.
2. In non-technical literature, it is also referred to as the *bell curve*.
3. Its CDF is closely related to the *error function*, erf(z).
4. The **FoldedNormal** and **HalfNormal** distributions are special cases.

Characterizations

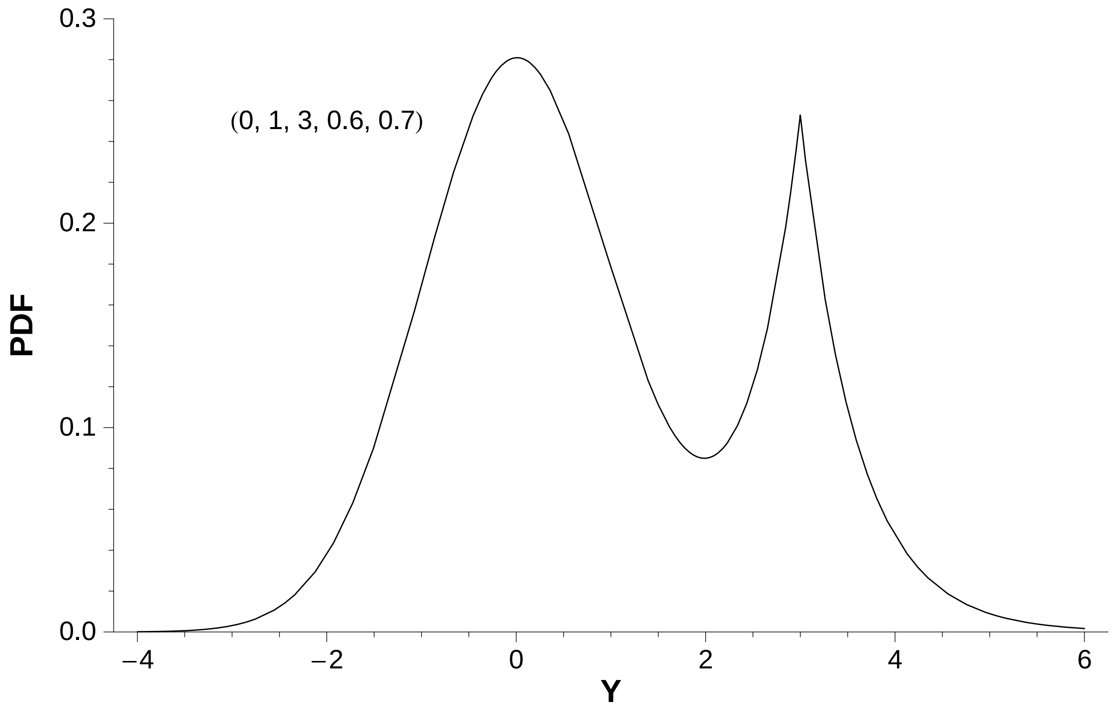
1. Let $Z_1, Z_2, \ldots, Z_N$ be i.i.d. random variables with finite values for their mean (μ) and variance (σ^2). Then, for any real number (z),

$$\lim_{N \to \infty} \text{Prob}\left[\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(\frac{Z_i - \mu}{\sigma}\right) \le z\right] = \Phi(z)$$

   known as the *Central Limit Theorem*. Loosely speaking, the sum of k random variables, from the same distribution, tends to be ~Normal, and more so as k increases.
2. If X ~Normal(A, B), then Y = a X + b ~Normal(a A + b, a B).
3. If X ~Normal(A, B) and Y ~Normal(C, D), then S = X + Y (i.e., the *convolution* of X and Y) is ~ $\text{Normal}\left(A + C, \sqrt{B^2 + D^2}\right)$.
4. Errors of real-valued observations are often ~Normal or ~**Laplace**.

Normal(A,B)&Laplace(C,D)

$B, D > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{B\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{(1-p)}{2D} \exp\left(-\frac{|y-C|}{D}\right)$$

$$\text{CDF} = p\,\Phi\left(\frac{y-A}{B}\right) + (1-p)\,\text{stdLaplaceCDF}\left(\frac{y-C}{D}\right)$$

Parameters -- A, C (μ_1, μ_2): Location, B, D (σ, λ): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = p\,A + (1-p)\,C$$

$$\text{Variance} = p\left[B^2 - (p-1)(A-C)^2\right] - 2(p-1)D^2$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.
3. The alternate, **Laplace(A,B)&Normal(C,D)** distribution may be obtained by switching identities in the parameter dialog. In this case, the parameters shown above in the Moments section must be reversed (cf. L&L and N&N).

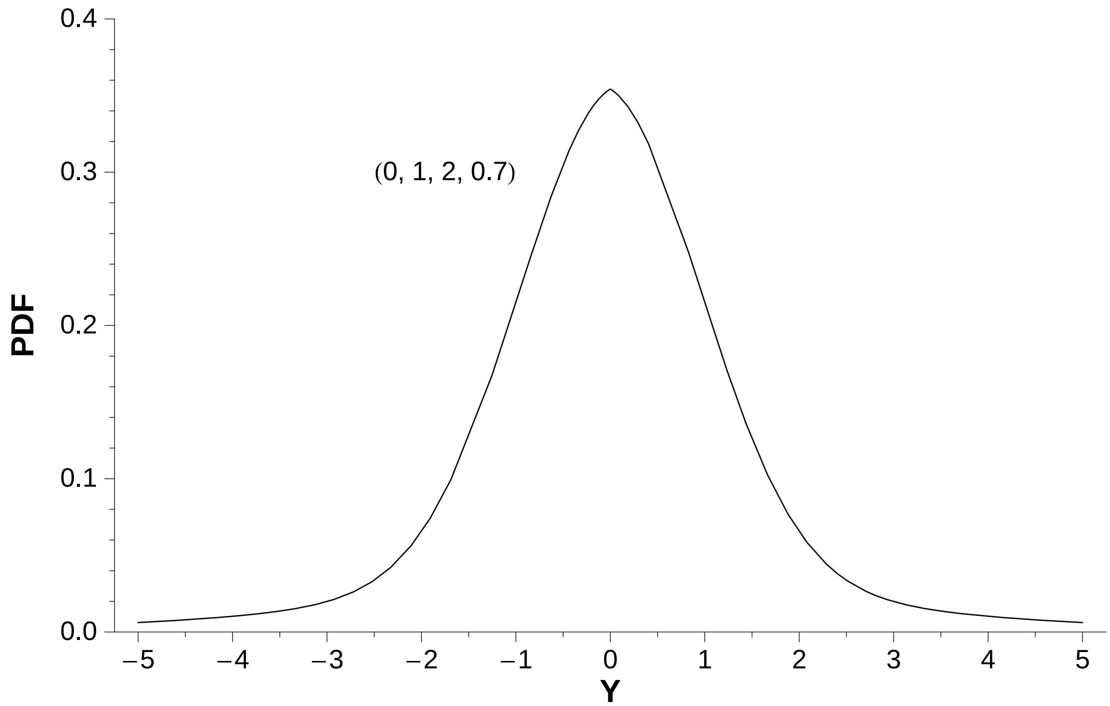
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Normal(A,B)&Laplace(A,C)

$B, C > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{B\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{(1-p)}{2C} \exp\left(-\frac{|y-A|}{C}\right)$$

$$\text{CDF} = p\,\Phi\left(\frac{y-A}{B}\right) + (1-p)\,\text{stdLaplaceCDF}\left(\frac{y-A}{C}\right)$$

Parameters -- A (μ): Location, B, C (σ, λ): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = p\,B^2 + 2\,(1-p)\,C^2$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = \frac{3\,p\,B^4 + 24\,(1-p)\,C^4}{\left(p\,B^2 + 2\,(1-p)\,C^2\right)^2} - 3$$

Q1, Q3: no simple closed form

$$\text{qMean} = \text{qMode} = 0.5$$

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.
3. The alternate, **Laplace(A,B)&Normal(A,C)** distribution may be obtained by switching identities in the parameter dialog. In this case, the parameters shown above in the Moments section must be reversed (cf. L&L and N&N).

Aliases and Special Cases

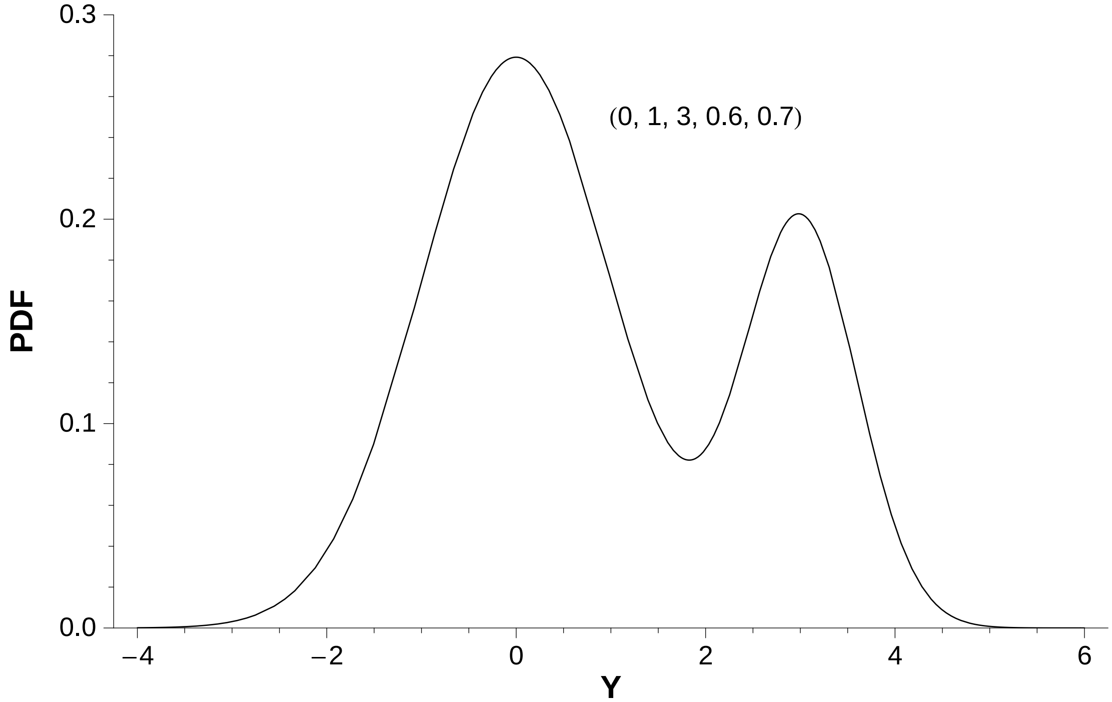
1. This is a special case of the **Normal(A,B)&Laplace(C,D)** distribution.

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Normal(A,B)&Normal(C,D)

B, D > 0,   0 < p < 1



$$\text{PDF} = \frac{p}{B\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{1-p}{D\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-C}{D}\right)^2\right)$$

$$\text{CDF} = p\,\Phi\left(\frac{y-A}{B}\right) + (1-p)\,\Phi\left(\frac{y-C}{D}\right)$$

Parameters -- A, C (μ_1, μ_2): Location, B, D (σ_1, σ_2): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = p\,A + (1-p)\,C$$

$$\text{Variance} = p\left[B^2 - (p-1)(A-C)^2\right] - (p-1)\,D^2$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.
3. Whether or not this much-studied mixture is bimodal depends partly upon parameter p. Obviously, if p is small enough, this mixture will be unimodal regardless of the remaining parameters. If

$$\left(A - C\right)^2 > \frac{8\, B^2\, D^2}{B^2 + D^2}$$

   then there will be some values of p for which this mixture is bimodal. However, if

$$\left(A - C\right)^2 < \frac{27\, B^2\, D^2}{4\left(B^2 + D^2\right)}$$

   then this mixture will be unimodal.

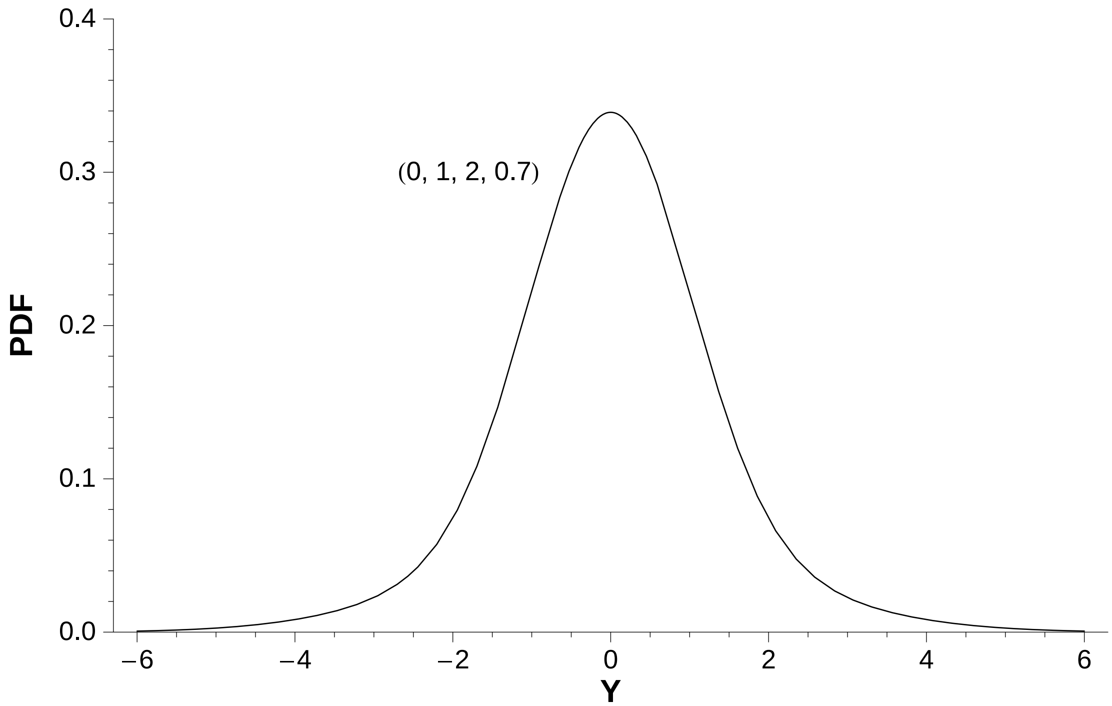
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Normal(A,B)&Normal(A,C)

$B, C > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{B\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{1-p}{C\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-A}{C}\right)^2\right)$$

$$\text{CDF} = p\,\Phi\left(\frac{y-A}{B}\right) + (1-p)\,\Phi\left(\frac{y-A}{C}\right)$$

Parameters -- A (μ): Location, B, C (σ_1, σ_2): Scale, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = p\,B^2 + (1-p)\,C^2$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = \frac{3\,p\,(1-p)\,(B^2 - C^2)}{\left(p\,B^2 + (1-p)\,C^2\right)^2}$$

Q1, Q3: no simple closed form

qMean = qMode = 0.5

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

Aliases and Special Cases

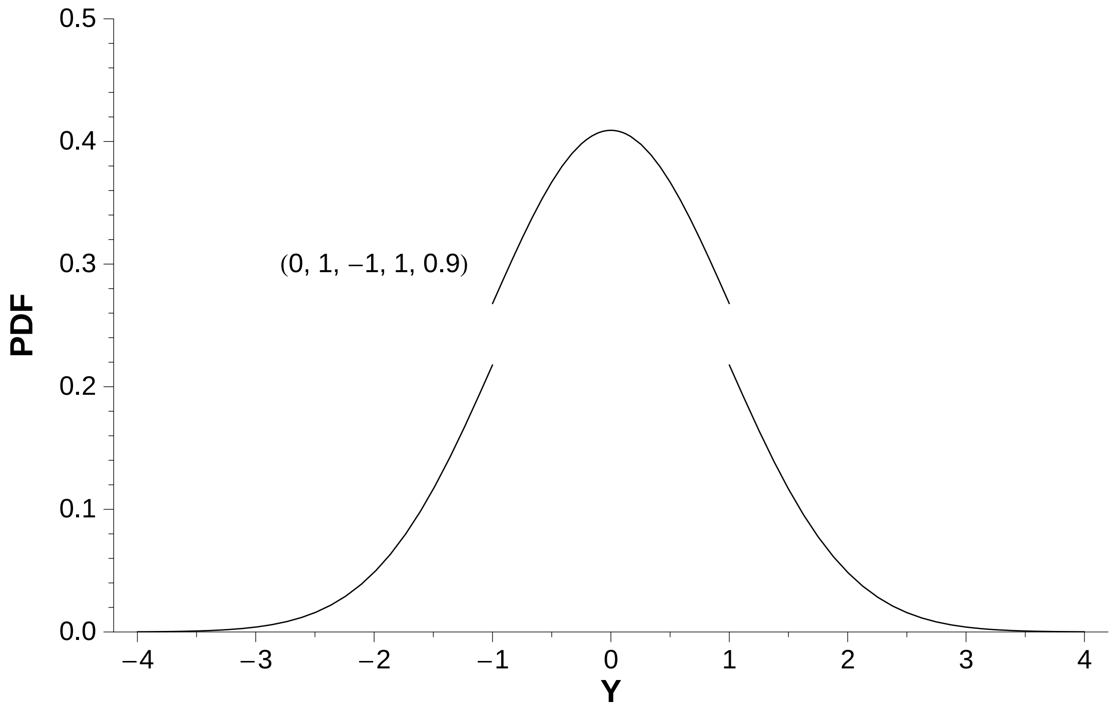
1. This is a special case of the **Normal(A,B)&Normal(C,D)** distribution.

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Normal(A,B)&Uniform(C,D)

$B > 0, \quad C < D, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{B\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{(1-p)\,(C<y<D)}{D-C}$$

$$\text{CDF} = \begin{cases} p\,\Phi\left(\frac{y-A}{B}\right), & y \le C \\ p\,\Phi\left(\frac{y-A}{B}\right) + (1-p)\left(\frac{y-C}{D-C}\right), & C < y < D \\ p\,\Phi\left(\frac{y-A}{B}\right) + (1-p), & y \ge D \end{cases}$$

Parameters -- A (μ), C: Location, B (σ), D : Scale (D = upper bound of Uniform(C,D)), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = p\,A + \frac{(1-p)\,(C+D)}{2}$$

$$\text{Variance} = p\,(A^2 + B^2) - \left[p\,A + \frac{(1-p)\,(C+D)}{2}\right]^2 + \frac{1}{3}(1-p)\,(C^2 + C\,D + D^2)$$

Quantiles, etc.: no simple closed form

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.

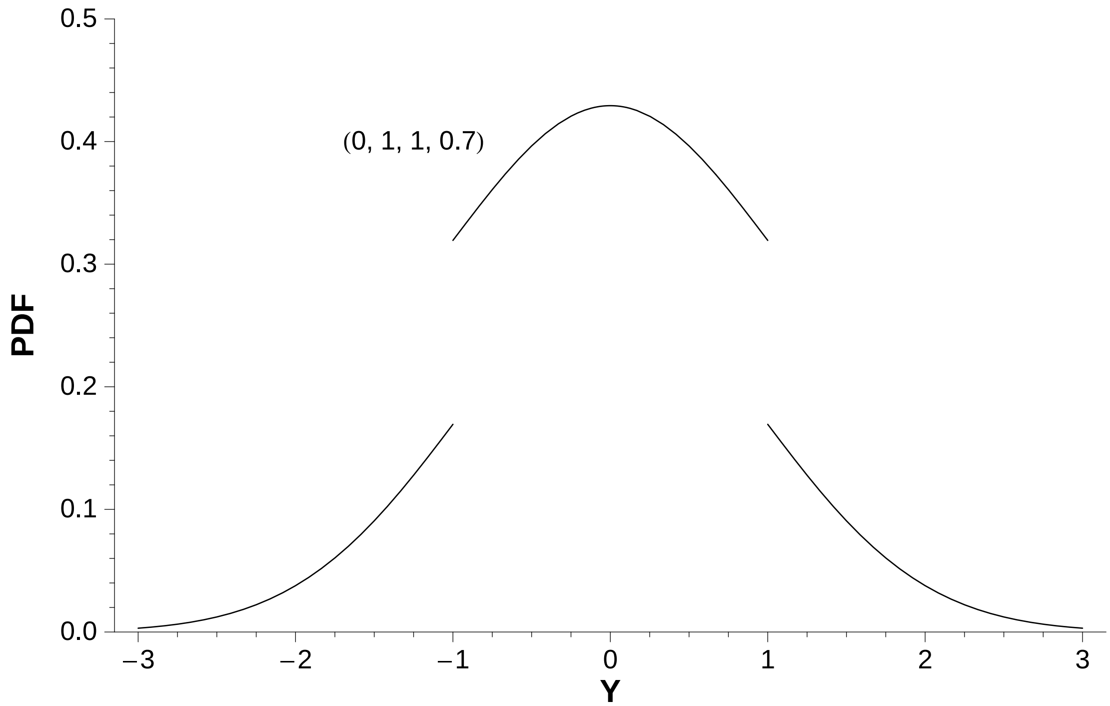
Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Normal(A,B)&Uniform(A,C)

$B, C > 0, \quad 0 < p < 1$



$$\text{PDF} = \frac{p}{B\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-A}{B}\right)^2\right) + \frac{(1-p)\left[(A-C) < y < (A+C)\right]}{2C}$$

$$\text{CDF} = \begin{cases} p\,\Phi\left(\frac{y-A}{B}\right), & y \le A-C \\ p\,\Phi\left(\frac{y-A}{B}\right) + (1-p)\left(\frac{y-A+C}{2C}\right), & A-C < y < A+C \\ p\,\Phi\left(\frac{y-A}{B}\right) + (1-p), & y \ge A+C \end{cases}$$

Parameters -- A (μ), C: Location, B (σ), C : Scale (C = half-width of Uniform(A,C)), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = \frac{3\,p\,B^2 + (1-p)\,C^2}{3}$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = \frac{9\left(15\,p\,B^4 + (1-p)\,C^4\right)}{5\left(3\,p\,B^2 + (1-p)\,C^2\right)^2} - 3$$

Q1, Q3: no simple closed form

qMean = qMode = 0.5

RandVar: determined by p

Notes

1. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
2. **Warning!** Mixtures usually have several local optima.
3. It is especially difficult to get the optimum value for parameter C in this distribution. There is an unusual amount of ambiguity in this case. It might be best to start out with parameter C Constant at its likely value (if known) and proceed from there.

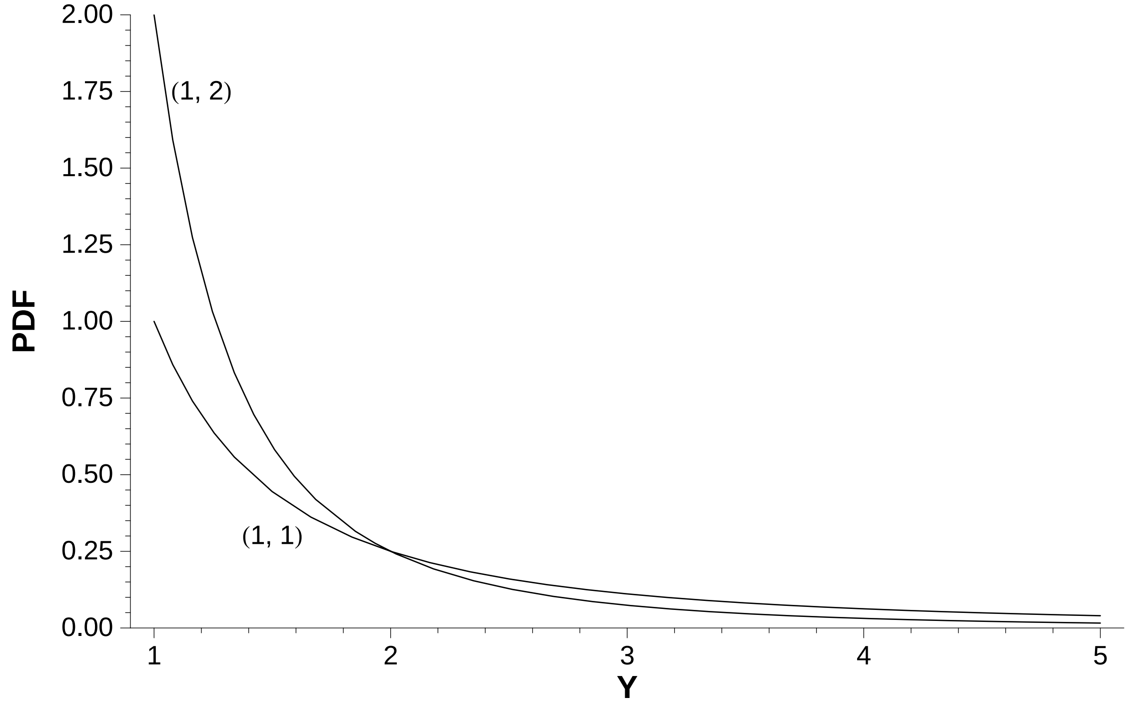
Aliases and Special Cases

1. This is a special case of the **Normal(A,B)&Uniform(C,D)** distribution.

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Pareto(A,B) $0 < A \le y, \quad B > 0$



$$\text{PDF} = \frac{B\,A^{B}}{y^{B+1}}$$

$$\text{CDF} = 1 - \left(\frac{A}{y}\right)^{B}$$

Parameters -- A (k): Location, Scale, B (a): Shape

Moments, etc. (see Note #3.)

$$\text{Mean} = \frac{A\,B}{B-1}$$

$$\text{Variance} = \frac{A^{2}\,B}{(B-2)\,(B-1)^{2}}$$

$$\text{Skewness} = \frac{2\left(B+1\right)\sqrt{B-2}}{\left(B-3\right)\sqrt{B}}$$

$$\text{Kurtosis} = \frac{6\left(B^{3}+B^{2}-6\,B-2\right)}{B\left(B^{2}-7\,B+12\right)}$$

$$\text{Mode} = A$$

$$\text{Median} = A\sqrt[B]{2}$$

$$Q1 = A\sqrt[B]{\frac{4}{3}} \qquad Q3 = A\sqrt[B]{4}$$

$$\text{qMean} = 1 - \left(\frac{B-1}{B}\right)^B \qquad \text{qMode} = 0$$

$$\text{RandVar} = \frac{A}{\sqrt[B]{u}}$$

Notes

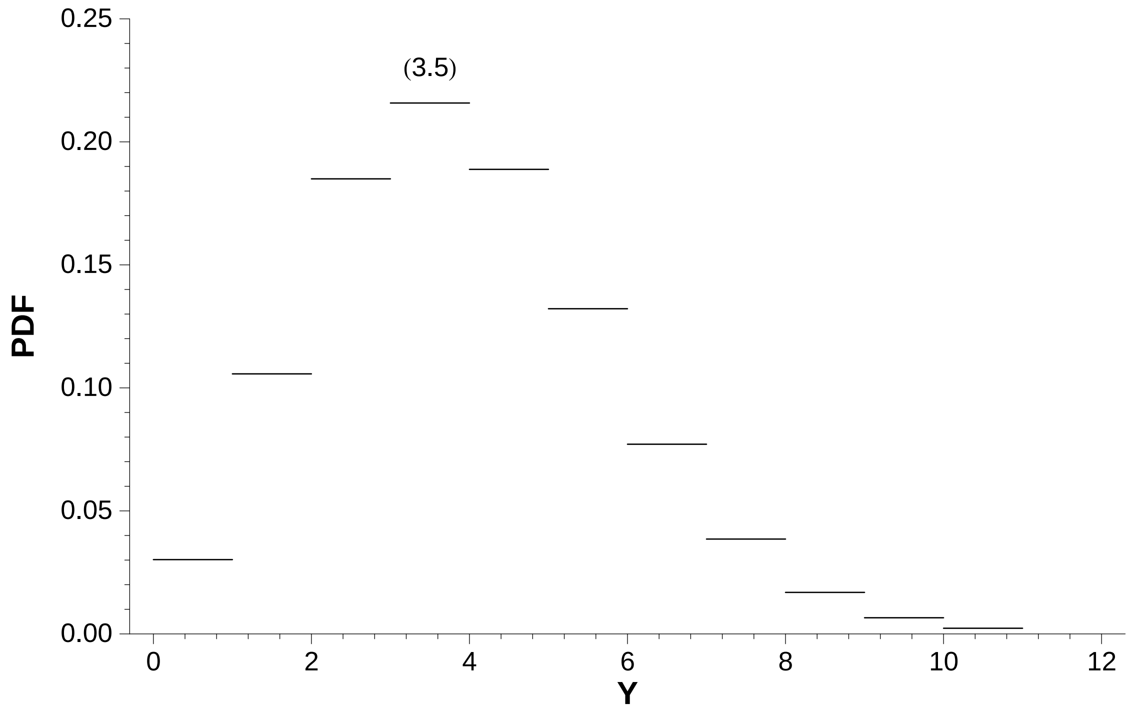
1. The Pareto distribution is always right-skewed.
2. There are several alternate forms for this distribution. In fact, the term Pareto is often applied to a class of distributions.
3. Moment k exists if $B > k$.

Aliases and Special Cases

Characterizations

1. The Pareto distribution is often used as an income distribution. Thus, the probability that a random income, in some defined population, exceeds a minimum, A, is ~Pareto.

Poisson(A) $y = 0, 1, 2, \ldots, \quad A > 0$



$$\text{PDF} = \frac{\exp\left(-A\right) A^{y}}{y!}$$

Parameters -- A (θ): Expectation

Moments, etc.

$$\text{Mean} = \text{Variance} = A$$

$$\text{Mode} = \text{int}\left(A\right)$$

Notes

1. If A is an integer, Mode also equals A – 1.

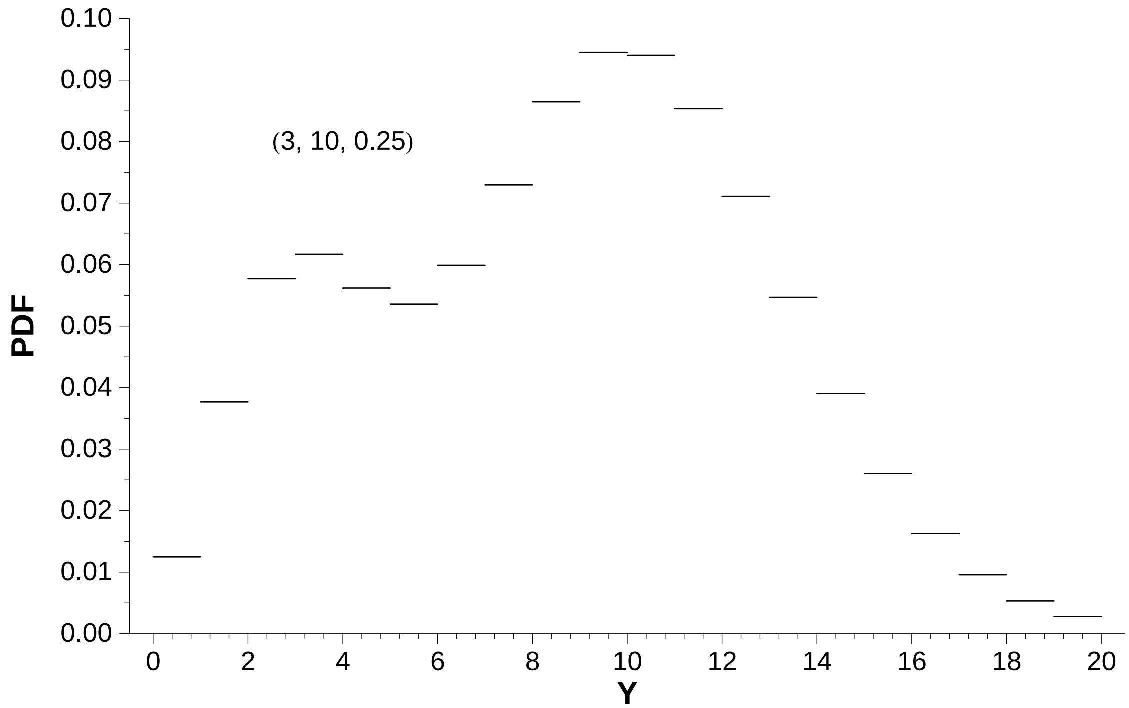
Aliases and Special Cases

Characterizations

1. The Poisson distribution is commonly used as an approximation to the **Binomial(A)** distribution when A is very small.
2. In queueing theory, when interarrival times are ~**Exponential**, the number of arrivals in a fixed interval are ~Poisson.
3. Errors in observations with integer values (i.e., miscounting) are ~Poisson.

Poisson(A)&Poisson(B)

$y = 0, 1, 2, \ldots, \quad 0 < A < B, \quad 0 < p < 1$



$$\text{PDF} = \frac{p \exp(-A)\, A^y + (1-p) \exp(-B)\, B^y}{y!}$$

Parameters -- A, B (θ_1, θ_2): Expectation, p: Weight of Component #1

Moments, etc.

$$\text{Mean} = p\, A + (1-p)\, B$$

$$\text{Variance} = p\, A\, (A + 1) + (1-p)\, B\, (B + 1) - \left[p\, (A - B) + B\right]^2$$

Mode: no simple closed form

Notes

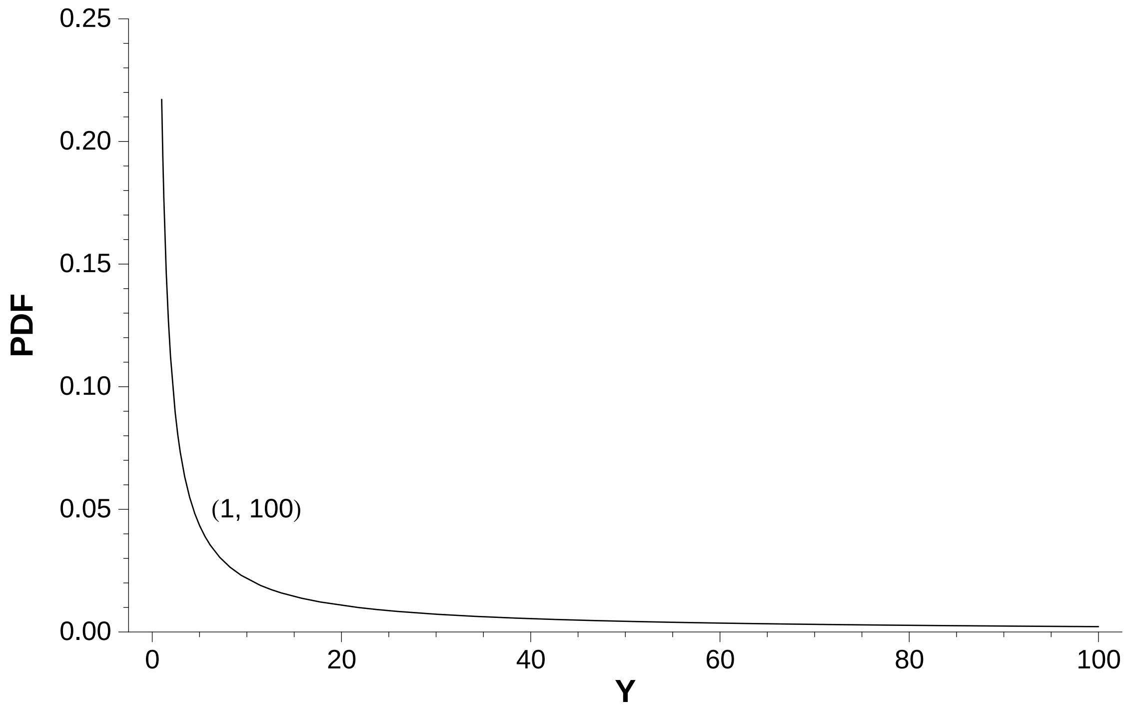
1. Here, parameter A is stipulated to be the Component with the smaller expectation.
2. This distribution may or may not be bimodal.
3. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
4. **Warning!** Mixtures usually have several local optima.

Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Reciprocal(A,B) $0 < A \le y \le B$



$$\text{PDF} = \frac{1}{y \log\left(\frac{B}{A}\right)}$$

$$\text{CDF} = \frac{\log\left(\frac{A}{y}\right)}{\log\left(\frac{A}{B}\right)}$$

Parameters -- A, B: Shape

Moments, etc. [$d \equiv \log(A/B)$]

$$\text{Mean} = \frac{A - B}{d}$$

$$\text{Variance} = \frac{(A - B)\left[A\,(d - 2) + B\,(d + 2)\right]}{2\,d^2}$$

$$\text{Skewness} = \frac{\sqrt{2}\left[12\,d\,(A-B)^2 + d^2\left(A^2\,(2\,d-9) + 2\,A\,B\,d + B^2\,(2\,d+9)\right)\right]}{3\,d\,\sqrt{A-B}\left[A\,(d-2) + B\,(d+2)\right]^{\frac{3}{2}}}$$

$$\text{Kurtosis} = \frac{-36\,(A-B)^3 + 36\,d\,(A-B)^2\,(A+B) - 16\,d^2\,(A^3-B^3) + 3\,d^3\,(A^2+B^2)\,(A+B)}{3\,(A-B)\left[A\,(d-2) + B\,(d+2)\right]^2} - 3$$

$$\text{Mode} = A$$

$$\text{Median} = \sqrt{A\,B}$$

$$\text{Q1} = \sqrt[4]{A^3}\,\sqrt[4]{B} \qquad \text{Q3} = \sqrt[4]{A}\,\sqrt[4]{B^3}$$

$$\text{qMean} = \frac{1}{d}\log\left(\frac{A\,d}{A-B}\right) \qquad \text{qMode} = 0$$

$$\text{RandVar} = A^{1-u}\,B^{u}$$

Notes

1. The Reciprocal distribution is unusual in that it has no location or scale parameter.

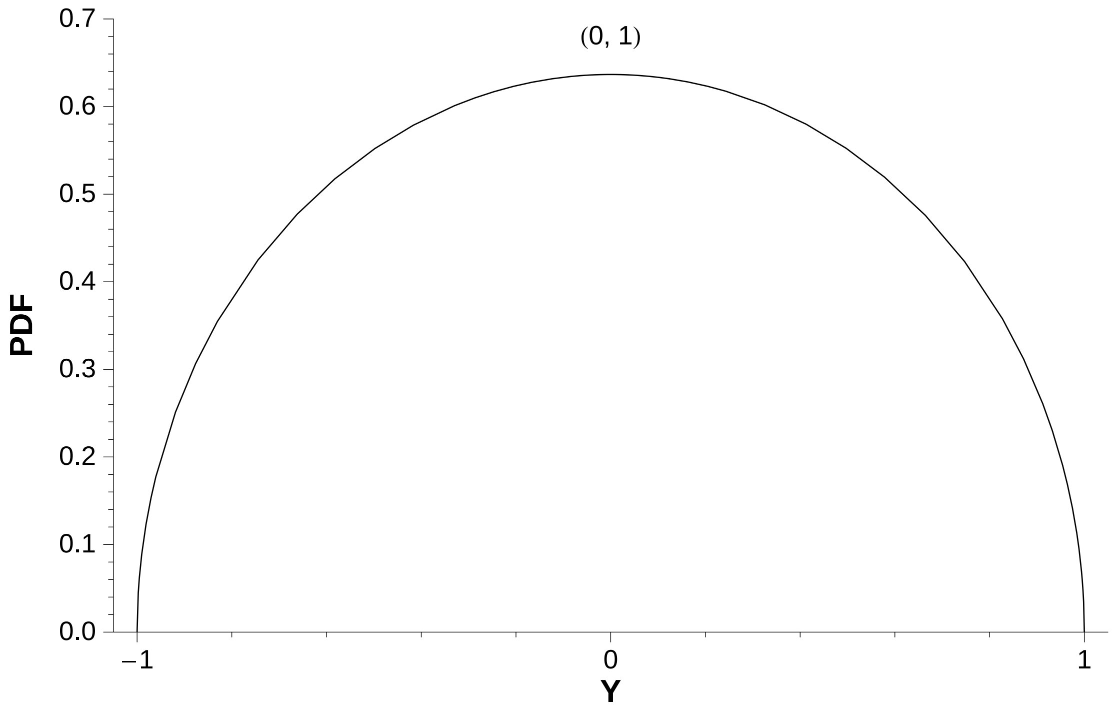
Aliases and Special Cases

Characterizations

1. The Reciprocal distribution is often used to describe *1/f noise*.

Semicircular(A,B)

$A - B \le y \le A + B, \quad B > 0$



$$\text{PDF} = \frac{2}{B\pi}\sqrt{1 - \left(\frac{y-A}{B}\right)^2}$$

$$\text{CDF} = \frac{1}{2} + \frac{1}{\pi}\left[\frac{y-A}{B}\sqrt{1 - \left(\frac{y-A}{B}\right)^2} + \operatorname{asin}\left(\frac{y-A}{B}\right)\right]$$

Parameters -- A: Location, B: Scale

Moments, etc.

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = \frac{B^2}{4}$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = -1$$

$$Q1 \approx A - 0.4040\,B \qquad Q3 \approx A + 0.4040\,B$$

$$\text{qMean} = \text{qMode} = 0.5$$

Notes

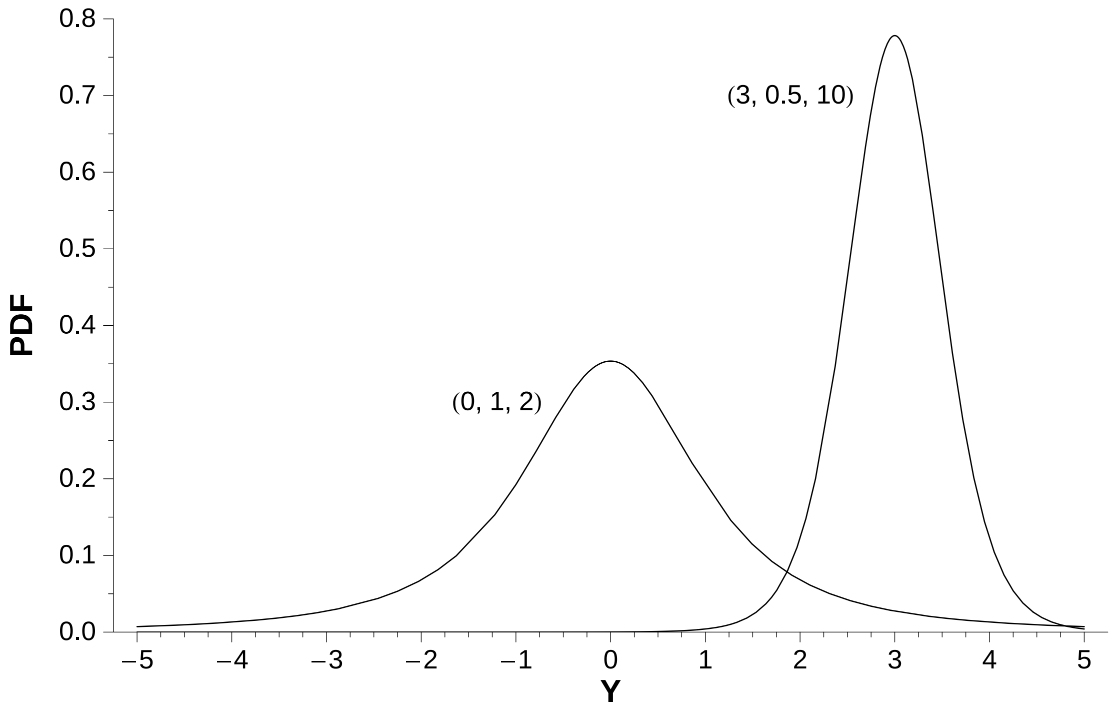
Aliases and Special Cases

Characterizations

1. The Semicircular distribution, like the **Cosine** distribution, is sometimes used as an alternative to the **Normal** distribution.

StudentsT(A,B,C)

$B > 0, \quad 0 < C \le 100$



$$\text{PDF} = \frac{\Gamma\left(\frac{C+1}{2}\right)}{B\sqrt{\pi C}\,\Gamma\left(\frac{C}{2}\right)}\left[1 + \frac{\left(\frac{y-A}{B}\right)^2}{C}\right]^{-\frac{C+1}{2}}$$

$$\text{CDF} = \begin{cases} \frac{1}{2}\, I\left(\frac{C}{C+t^2}, \frac{C}{2}, \frac{1}{2}\right), & t \equiv \frac{y-A}{B} \le 0 \\ 1 - \frac{1}{2}\, I\left(\frac{C}{C+t^2}, \frac{C}{2}, \frac{1}{2}\right), & t \equiv \frac{y-A}{B} > 0 \end{cases}$$

Parameters -- A: Location, B: Scale, C (ν): Shape (also, degrees of freedom)

Moments, etc. (See Note #4.)

$$\text{Mean} = \text{Median} = \text{Mode} = A$$

$$\text{Variance} = \left(\frac{C}{C-2}\right) B^2$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = 3\left[\frac{(C-2)^2\,\Gamma\left(\frac{C}{2}-2\right)}{4\,\Gamma\left(\frac{C}{2}\right)} - 1\right]$$

Q1, Q3: no simple closed form

$$\text{qMean} = \text{qMode} = 0.5$$

Notes

1. In the literature, C > 0. The bounds shown here are used to prevent divergence.
2. The Student's-t distribution approaches the **Normal** distribution asymptotically as C -> infinity.
3. If the optimum model has C = 100, use **Normal** instead.
4. Moment k exists if C > k.

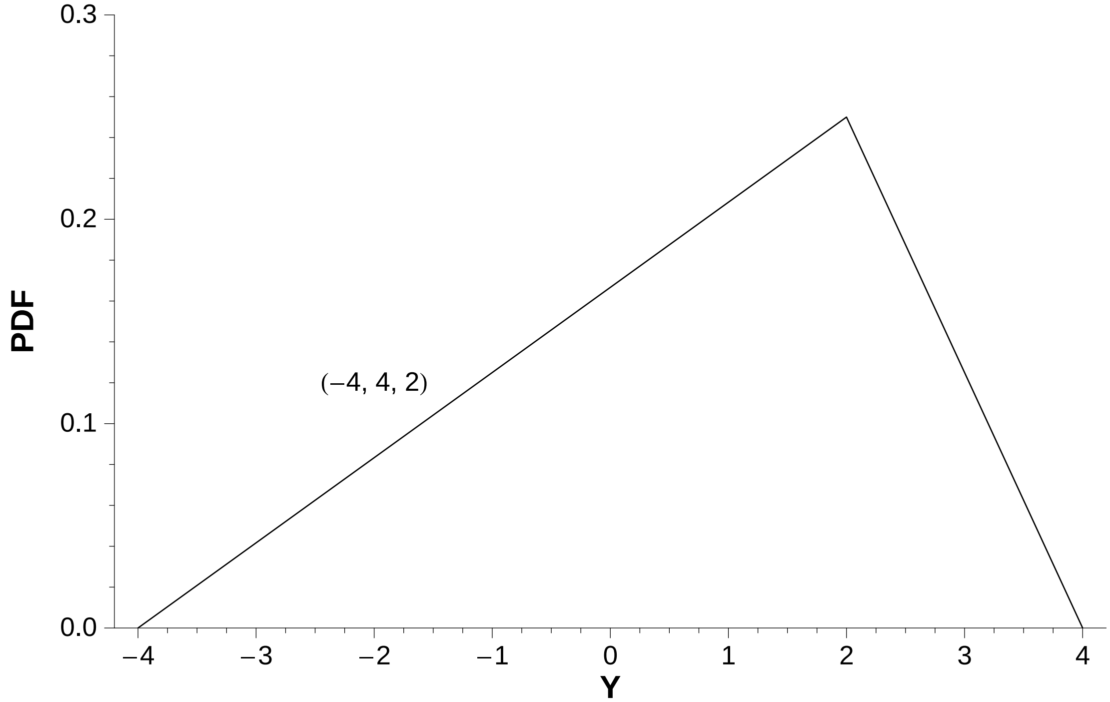
Aliases and Special Cases

1. The Student's-t distribution is often referred to as simply the *t-distribution*.

Characterizations

1. The Student's-t distribution is used to characterize small samples (typically, N < 30) from a **Normal** population.

Triangular(A,B,C) A < y, C < B



$$\text{PDF} = \begin{cases} \dfrac{2\left(y-A\right)}{\left(B-A\right)\left(C-A\right)}, & y < C \\ \dfrac{2\left(B-y\right)}{\left(B-A\right)\left(B-C\right)}, & y \geq C \end{cases}$$

$$\text{CDF} = \begin{cases} \dfrac{\left(y-A\right)^2}{\left(B-A\right)\left(C-A\right)}, & y < C \\ \dfrac{A\left(B-C\right) + B\left(C-2\,y\right) + y^2}{\left(A-B\right)\left(B-C\right)}, & y \geq C \end{cases}$$

Parameters -- A: Location, B: Scale (upper bound), C: Shape (Mode)

Moments, etc. $\left[d \equiv A^2 + B^2 - B\,C + C^2 - A\left(B+C\right) \right]$

$$\text{Mean} = \frac{A+B+C}{3}$$

$$\text{Variance} = \frac{d}{18}$$

$$\text{Skewness} = \frac{\sqrt{2}\,(A+B-2\,C)\,(2\,A-B-C)\,(A-2\,B+C)}{5\,\sqrt{d^3}}$$

$$\text{Kurtosis} = -\frac{3}{5}$$

$$\text{Mode} = C$$

$$\text{Median} = A + \frac{1}{\sqrt{2}}\sqrt{(B-A)\,(C-A)}\;,\;\text{if } C \geq \frac{A+B}{2} \text{ else } B - \frac{1}{\sqrt{2}}\sqrt{(B-A)\,(B-C)}$$

$$\text{Q1} = A + \frac{1}{2}\sqrt{(B-A)\,(C-A)}\;,\;\text{if qMode} \geq \frac{1}{4} \text{ else } B - \frac{1}{2}\sqrt{3\,(B-A)\,(B-C)}$$

$$\text{Q3} = A + \frac{1}{2}\sqrt{3\,(B-A)\,(C-A)}\;,\;\text{if qMode} \geq \frac{3}{4} \text{ else } B - \frac{1}{2}\sqrt{(B-A)\,(B-C)}$$

$$\text{qMean} = \begin{cases} \dfrac{(B+C-2\,A)^2}{9\,(B-A)\,(C-A)}\;, & C \geq \dfrac{A+B}{2} \\ \dfrac{A^2+5\,A\,B-5\,B^2-7\,A\,C+5\,B\,C+C^2}{9\,(A-B)\,(B-C)}\;, & C < \dfrac{A+B}{2} \end{cases}$$

$$\text{qMode} = \frac{C-A}{B-A}$$

$$\text{RandVar} = A + \sqrt{u\,(B-A)\,(C-A)}\;,\;\text{if qMode} \geq u \text{ else } B - \sqrt{(1-u)\,(B-A)\,(B-C)}$$

Notes

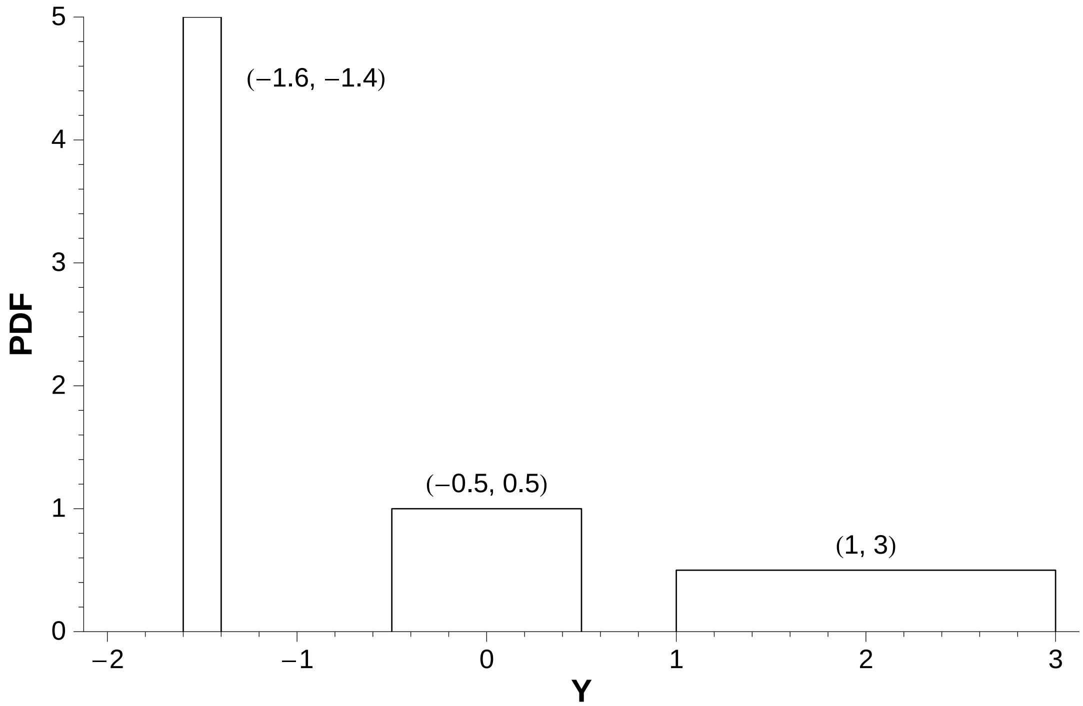
Aliases and Special Cases

Characterizations

1. If X is ~**Uniform**(a,b) and Z is ~**Uniform**(c,d) and (b – a) = (d – c), then (X + Z) is ~Triangular(a+c,b+d,(a+b+c+d)/2). The latter is called the *convolution* of the two **Uniform** distributions.

Uniform(A,B)

$A < y < B$



$$\text{PDF} = \frac{1}{B - A}$$

$$\text{CDF} = \frac{y - A}{B - A}$$

Parameters -- A : Location, B : Scale (upper bound)

Moments, etc.

$$\text{Mean} = \text{Median} = \frac{A + B}{2}$$

$$\text{Variance} = \frac{1}{12}\left(B - A\right)^2$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = -\frac{6}{5}$$

$$\text{Mode} = \text{none}$$

$$Q1 = \frac{3\,A + B}{4} \qquad Q3 = \frac{A + 3\,B}{4}$$

$$qMean = 0.5$$

$$RandVar = A + u\left(B - A\right)$$

Notes

1. The parameters for the Uniform distribution are sometimes given, equivalently, as the mean and half-width of the domain.

Aliases and Special Cases

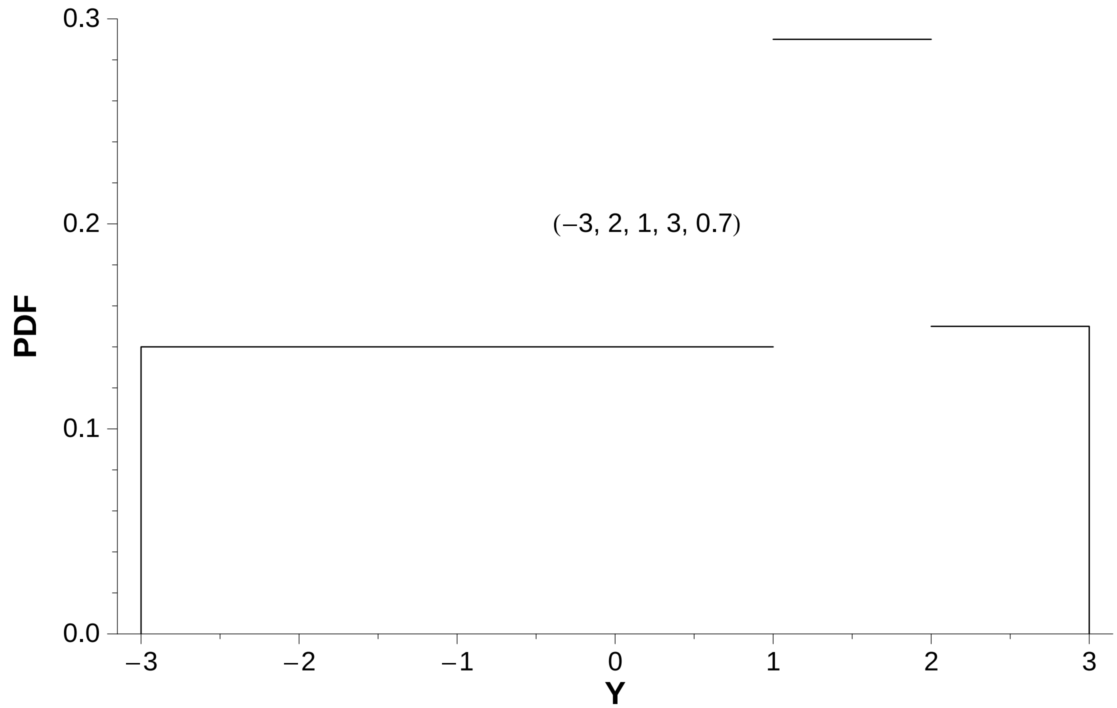
1. The Uniform distribution is often called the *Rectangular* distribution.

Characterizations

1. The Uniform distribution is commonly used to describe total ignorance within a bounded interval.
2. Pseudo-random number generators typically return X ~Uniform(0, 1) as their primary output.

Uniform(A,B)&Uniform(C,D)

$A < y < B \text{ or } D, \quad A < C < B, D, \quad 0 < p < 1$



$$\text{PDF} = \frac{p\,(y < B)}{B - A} + \frac{(1 - p)\,(C < y < D)}{D - C}$$

$$\text{CDF} = \text{cdf1} + \text{cdf2}$$

where

$$\text{cdf1} = p \text{ if } (y > B), \text{ else } \frac{p\,(y - A)}{B - A} \text{ and}$$

$$\text{cdf2} = 0 \text{ if } (y < C), \text{ else } (1 - p) \text{ if } (y > D), \text{ else } \frac{(1 - p)\,(y - C)}{D - C}$$

Parameters -- A, C: Location, B, D: Scale (upper bounds), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \frac{p\,(A + B) + (1 - p)\,(C + D)}{2}$$

$$\text{Variance} = \frac{p\left(A^2 + A\,B + B^2\right) - \frac{3}{4}\left[p\,(A + B) + (1 - p)\,(C + D)\right]^2 + (1 - p)\left(C^2 + C\,D + D^2\right)}{3}$$

Quantiles, etc.: case-dependent

RandVar: determined by p

Notes

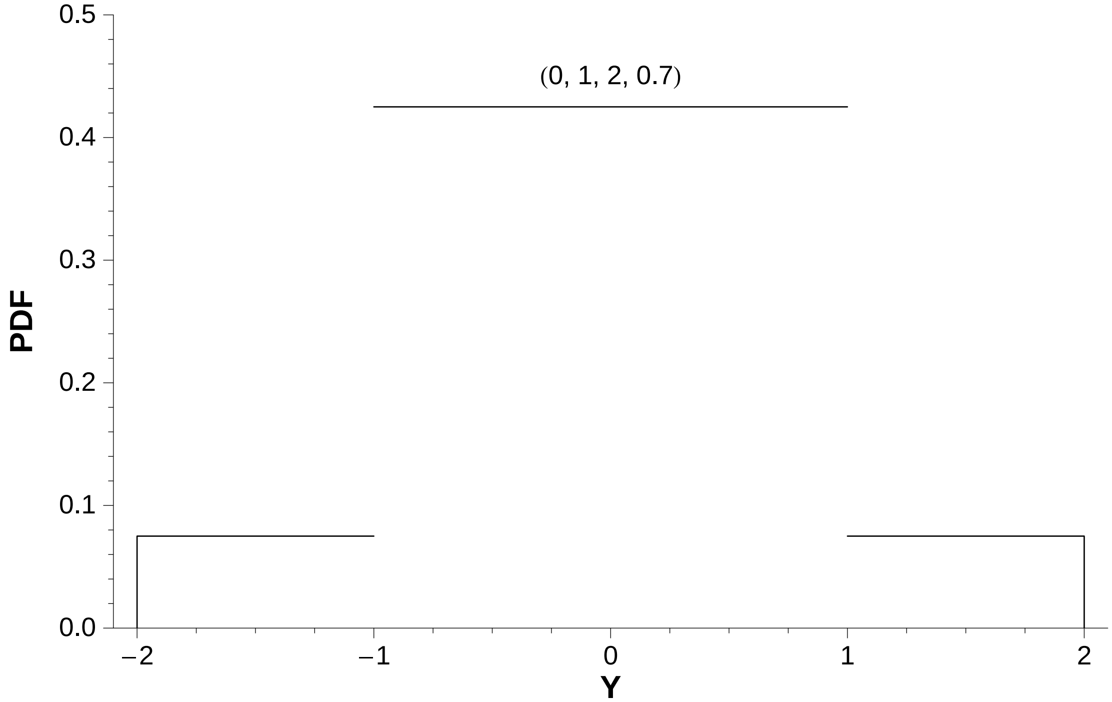
1. As implemented here, this distribution requires overlap between the two components. Also, Component #1 must cover min(y) although either Component may cover max(y).
2. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
3. **Warning!** Mixtures usually have several local optima.

Aliases and Special Cases

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Uniform(A,B)&Uniform(A,C) $A - C < y < A + C, \quad B < C, \quad 0 < p < 1$



$$\text{PDF} = \frac{p\left[\left(A - B\right) < y < \left(A + B\right)\right]}{2\,B} + \frac{\left(1 - p\right)}{2\,C}$$

$$\text{CDF} = \begin{cases} \dfrac{1-p}{2\,C}\left(y - A + C\right), & y \le A - B \\ \dfrac{p}{2\,B}\left(y - A + B\right) + \dfrac{1-p}{2\,C}\left(y - A + C\right), & A - B < y < A + B \\ p + \dfrac{1-p}{2\,C}\left(y - A + C\right), & y \ge A + B \end{cases}$$

Parameters -- A: Location, B, C: Scale (half-widths), p: Weight of Component #1

Moments, etc.

$$\text{Mean} = \text{Median} = A$$

$$\text{Variance} = \frac{p\,B^2 + \left(1 - p\right)C^2}{3}$$

$$\text{Skewness} = 0$$

$$\text{Kurtosis} = \frac{9\left(p\,B^4 + \left(1-p\right)C^4\right)}{5\left(p\,B^2 + \left(1-p\right)C^2\right)^2} - 3$$

$$\text{Mode} = \text{none}$$

Q1, Q3: determined by p

$$\text{qMean} = 0.5$$

RandVar: determined by p

Notes

1. Note that the parameters are defined differently here than in all other Uniform distributions discussed elsewhere in this document.
2. Binary mixtures may require hundreds of data points for adequate optimization and, even then, often have unusually wide confidence intervals. In fact, the criterion response is sometimes flat over a broad range, esp. with respect to parameter p.
3. **Warning!** Mixtures usually have several local optima.

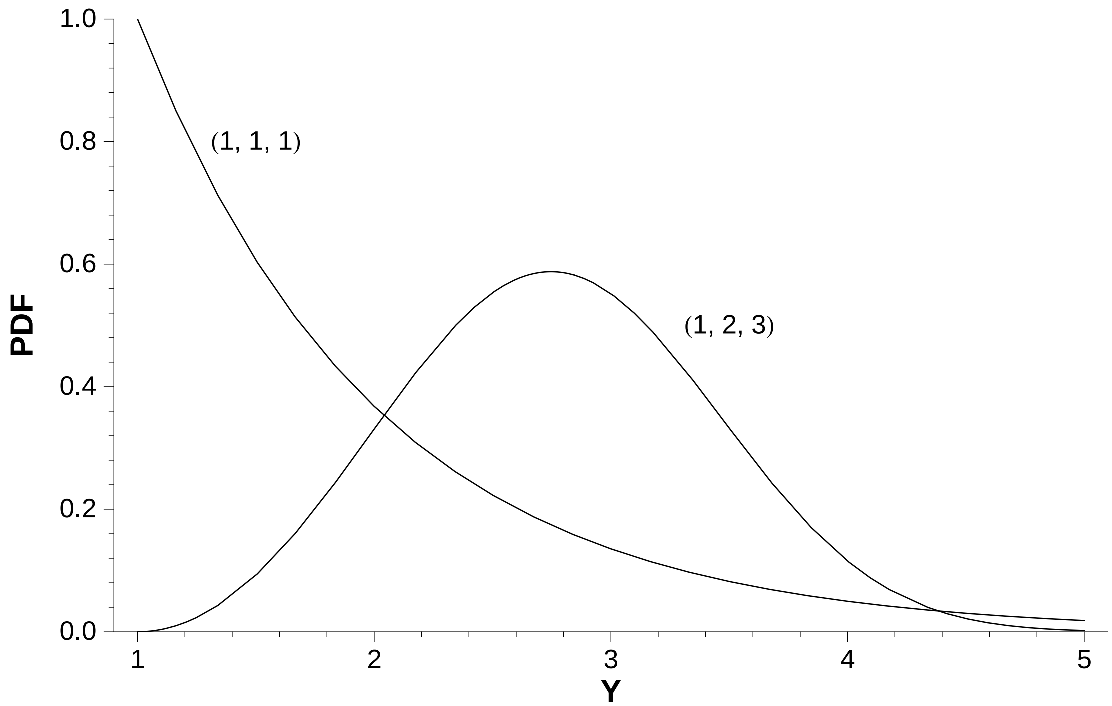
Aliases and Special Cases

1. This is a special case of the **Uniform(A,B)&Uniform(C,D)** distribution.

Characterizations

1. The usual interpretation of a binary mixture is that it represents an undifferentiated composite of two populations having parameters and weights as described above.

Weibull(A,B,C) $y > A, \quad B, C > 0$



$$\text{PDF} = \frac{C}{B}\left(\frac{y-A}{B}\right)^{C-1} \exp\left(-\left(\frac{y-A}{B}\right)^{C}\right)$$

$$\text{CDF} = 1 - \exp\left(-\left(\frac{y-A}{B}\right)^{C}\right)$$

Parameters -- A (ξ): Location, B (α): Scale, C (c): Shape

Moments, etc.

$$\text{Mean} = A + B\,\Gamma\left(\frac{C+1}{C}\right)$$

$$\text{Variance} = B^2\left[\Gamma\left(\frac{C+2}{C}\right) - \Gamma^2\left(\frac{C+1}{C}\right)\right]$$

$$\text{Skewness} = \frac{2\,\Gamma^3\left(\frac{C+1}{C}\right) - 3\,\Gamma\left(\frac{C+1}{C}\right)\Gamma\left(\frac{C+2}{C}\right) + \Gamma\left(\frac{C+3}{C}\right)}{\sqrt{\left(\Gamma\left(\frac{C+2}{C}\right) - \Gamma^2\left(\frac{C+1}{C}\right)\right)^3}}$$

$$\text{Kurtosis} = \frac{-3\,\Gamma^4\left(\frac{C+1}{C}\right) + 6\,\Gamma^2\left(\frac{C+1}{C}\right)\Gamma\left(\frac{C+2}{C}\right) - 4\,\Gamma\left(\frac{C+1}{C}\right)\Gamma\left(\frac{C+3}{C}\right) + \Gamma\left(\frac{C+4}{C}\right)}{\left(\Gamma^2\left(\frac{C+1}{C}\right) - \Gamma\left(\frac{C+2}{C}\right)\right)^2} - 3$$

$$\text{Mode} = \begin{cases} A\,, & C \le 1 \\ A + B\sqrt[C]{\frac{C-1}{C}}\,, & C > 1 \end{cases}$$

$$\text{Median} = A + B\sqrt[C]{\log\left(2\right)}$$

$$\text{Q1} = A + B\sqrt[C]{\log\left(\frac{4}{3}\right)} \qquad \text{Q3} = A + B\sqrt[C]{\log\left(4\right)}$$

$$\text{qMean} = 1 - \exp\left(-\Gamma^C\left(\frac{C+1}{C}\right)\right) \qquad \text{qMode} = 1 - \exp\left(\frac{1-C}{C}\right),\ C > 1;\ \text{else } 0$$

$$\text{RandVar} = A + B\sqrt[C]{-\log\left(u\right)}$$

Notes

1. The Weibull distribution is roughly symmetrical for C near 3.6. When C is smaller/larger, the distribution is left/right-skewed, respectively.

Aliases and Special Cases

1. The Weibull is sometimes known as the *Frechet* distribution.
2. Weibull(–y) is the upper-bound analogue of the **ExtremeLB** distribution, one of a class of *Extreme-value* distributions.
3. Weibull(A,B,1) is the **Exponential** distribution.
4. Weibull(0,1,2) is the standard *Rayleigh* distribution.

Characterizations

1. If $X = \left(\frac{y-A}{B}\right)^C$ is ~ standard **Exponential**, then y ~Weibull(A, B, C). Thus, the Weibull distribution is a generalization of the **Exponential** distribution.
2. Weibull(–y) is an extreme-value distribution for variates with an upper bound.